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The Multiplication Operator in Sobolev Spaces with Respect to Measures

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We consider the multiplication operator, M, in Sobolev spaces with respect to general measures and give a characterization for M to be bounded, in terms of sequentially dominated measures. This has important consequences for the asymptotic behaviour of Sobolev orthogonal polynomials. Also, we study properties of Sobolev spaces with respect to measures.

Key Words: Sobolev spaces; weights; orthogonal polynomials.

1. INTRODUCTION

Weighted Sobolev spaces are an interesting topic in many fields of mathematics (see, e.g., [HKM, K, Ku, KO, KS, T]. In [ELW1, EL, ELW2] the authors study some examples of Sobolev spaces with respect to general measures instead of weights, in relation with ordinary differential equations and Sobolev orthogonal polynomials. The papers [RARP1, RARP2] are the beginning of a theory of Sobolev spaces with respect to general measures. We are interested in the relationship between this topic and Sobolev orthogonal polynomials.

Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ a vectorial Borel measure in $\mathbf{R}$ with $\Delta := \bigcup_{j=0}^{k} \operatorname{supp} \mu_j$. The Sobolev norm of a function f of class $C^k(\mathbf{R})$ in $W^{k,p}(\Delta, \mu)$ is defined by

$$\|f\|^p_{W^{k,p}(\Delta,\mu)} := \sum_{j=0}^{k} \int |f^{(j)}|^p \, d\mu_j .$$

We talk about Sobolev norm although it can be a seminorm; in this case we will take equivalence classes, as usual.

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We say that $\mu \in \mathcal{M}$ if every polynomial belongs to $L^1(\mu_0) \cap L^1(\mu_1) \cap \cdots \cap L^1(\mu_k)$. Therefore if $\mu \in \mathcal{M}$, every polynomial belongs to $L^p(\mu_0) \cap L^p(\mu_1) \cap \cdots \cap L^p(\mu_k)$ for any $1 \leqslant p < \infty$. Obviously, every $\mu \in \mathcal{M}$ is finite. If Δ is a compact set, we have $\mu \in \mathcal{M}$ if and only if μ is finite. If $\mu \in \mathcal{M}$, we denote by $P^{k,p}(\Delta, \mu)$ the completion of polynomials P with the norm of $W^{k,p}(\Delta, \mu)$. By a theorem in [LP] we know that the zeros of the Sobolev orthogonal polynomials with respect to the scalar product in $W^{k,2}(\Delta, \mu)$ are contained in the disk $\{z: |z| \leqslant 2\, \|M\|\}$, where the multiplication operator $(Mf)(x) = x\, f(x)$ is considered in the space $P^{k,2}(\Delta, \mu)$. Consequently, the set of the zeros of the Sobolev orthogonal polynomials is bounded if the multiplication operator is bounded. The location of these zeros allows to prove results on the asymptotic behaviour of Sobolev orthogonal polynomials (see [LP]).

In [LP] also appears the following result: If Δ is a compact set and μ is a finite measure in Δ sequentially dominated, then M is a bounded operator in $P^{k,2}(\Delta, \mu)$, where the vectorial measure μ is sequentially dominated if $\#\operatorname{supp} \mu_0 = \infty$ and $d\mu_j = f_j\, d\mu_{j-1}$ with f_j bounded for $1 \leqslant j \leqslant k$. In that paper the authors ask for other conditions on M to be bounded.

It is not difficult to see that the multiplication operator can be bounded when the vectorial measure is not sequentially dominated. In Section 4 below and in [RARP2] other conditions are given in order to have the boundedness of M.

Now, let us state the main results here. We refer to the definitions in Sections 2 and 4. In the paper, the results are numbered according to the section where they are proved.

Here we obtain the following characterization for the boundedness of the multiplication operator in terms of comparable norms. Observe that this characterization is closely related to sequentially dominated measures, since we say that a vectorial measure μ belongs to the class ESD (extended sequentially dominated) if and only if $d\mu_j = f_j\, d\mu_{j-1}$ with f_j bounded for $1 \leqslant j \leqslant k$.

THEOREM 4.1. *Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ a finite vectorial measure with Δ a compact set. Then, the multiplication operator is bounded in $P^{k,p}(\Delta, \mu)$ if and only if there exists a vectorial measure $\mu' \in ESD$ such that the Sobolev norms in $W^{k,p}(\Delta, \mu)$ and $W^{k,p}(\Delta, \mu')$ are comparable on P. Furthermore, we can choose $\mu' = (\mu'_0, ..., \mu'_k)$ with $\mu'_j := \mu_j + \mu_{j+1} + \cdots + \mu_k$.*

We have also necessary conditions and sufficient conditions for M to be bounded. The following are the most important.

THEOREM 4.3. *Let us consider $1 \leqslant p < \infty$ and a finite vectorial measure μ with Δ a compact set. Assume that $(\Delta^{ad}, \mu^{ad}) \in \mathscr{C}_0$ and that for each $1 \leqslant j \leqslant k$ we have $\mu_j(\Delta \setminus (J_{j-1} \cup K_{j-1})) = 0$, where K_{j-1} is a finite union of compact intervals contained in $\Omega^{(j-1)}$, and J_{j-1} is a measurable set with $d\mu_j = f_j \, d\mu_{j-1}$ in J_{j-1} and f_j bounded. Then the multiplication operator is bounded in $P^{k,p}(\Delta, \mu)$.*

THEOREM 4.4. *Let us consider $1 < p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ a finite vectorial measure such that Δ is a compact set and there exist $\eta_0 > 0$, $x_0 \in \mathbf{R}$ and $0 < k_0 \leqslant k$ with $\mu_j([x_0 - \eta_0, x_0 + \eta_0]) = 0$ for $k_0 < j \leqslant k$. Let us assume that x_0 is neither right nor left $(k_0 - 1)$-regular. If $\mu_{k_0}(\{x_0\}) > 0$ and $\mu_{k_0 - 1}(\{x_0\}) = 0$, then the multiplication operator is not bounded in $P^{k,p}(\Delta, \mu)$.*

THEOREM 4.5. *Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ a finite vectorial measure with Δ a compact set. Assume that there exist $x_0 \in \mathbf{R}$, $c > 0$, $0 \leqslant k_0 < k$ and an open neighbourhood U of x_0 such that*

$$d\mu_{j+1}(x) \leqslant c \, |x - x_0|^p \, d\mu_j(x),$$

for $x \in U \setminus \{x_0\}$ and $k_0 \leqslant j < k$. If there exists $i > k_0$ with $\mu_i(\{x_0\}) > 0$ and $\mu_{i-1}(\{x_0\}) = 0$, then the multiplication operator is not bounded in $P^{k,p}(\Delta, \mu)$.

In order to prove these results we also obtain some results on Sobolev spaces with respect to measures, which are interesting by themselves.

THEOREM 3.1. *Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ a finite vectorial measure with Δ a compact set. Assume that there exist $x_0 \in \mathbf{R}$ and $0 \leqslant k_0 \leqslant k$ with $\mu_{k_0}(\{x_0\}) = 0$ and satisfying the following property if $k_0 < k$: there exist an open neighbourhood U of x_0 and $c > 0$ such that*

$$d\mu_{j+1}(x) \leqslant c \, |x - x_0|^p \, d\mu_j(x),$$

for $x \in U$ and $k_0 \leqslant j < k$. Let us define

$$v := (0, ..., 0, \alpha_{k_0} \delta_{x_0}, \alpha_{k_0+1} \delta_{x_0}, ..., \alpha_k \delta_{x_0})$$

and $N := \#\{k_0 \leqslant j \leqslant k : \alpha_j > 0\}$. Given a Cauchy sequence $\{q_n\} \subset P$ in $W^{k,p}(\Delta, \mu)$ and $u_{k_0}, ..., u_k \in \mathbf{R}$ there exists a Cauchy sequence $\{r_n\} \subset P$ in $W^{k,p}(\Delta, \mu)$ with $\lim_{n \to \infty} \|q_n - r_n\|_{W^{k,p}(\Delta, \mu)} = 0$ and $r_n^{(j)}(x_0) = u_j$ for $k_0 \leqslant j \leqslant k$. Consequently $P^{k,p}(\Delta, \mu + v)$ is isomorphic to $P^{k,p}(\Delta, \mu) \times \mathbf{R}^N$.

THEOREM 3.3. *Let us consider $1 < p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ a vectorial measure such that there exist $\eta_0 > 0$, $x_0 \in \mathbf{R}$ and $0 < k_0 \leqslant k$ with $\mu_j([x_0 - \eta_0, x_0 + \eta_0]) < \infty$ for $0 \leqslant j \leqslant k_0$ and $\mu_j([x_0 - \eta_0, x_0 + \eta_0]) = 0$ for*

$k_0 < j \leqslant k$ (if $k_0 < k$). Let us assume that x_0 is neither right nor left (k_0-1)-regular and that $\mu_{k_0-1}(\{x_0\})=0$. Then, for any $0<\eta\leqslant\eta_0$, there is no positive constant c_1 with

$$c_1\,|f^{(k_0-1)}(x_0)| \leqslant \|f\|_{W^{k,p}(\Delta,\mu)},$$

for every $f \in C_c^\infty([x_0-\eta, x_0+\eta])$.

If we have also that μ is finite and Δ is a compact set, then there is no positive constant c_2 with

$$c_2\,|q^{(k_0-1)}(x_0)| \leqslant \|q\|_{W^{k,p}(\Delta,\mu)},$$

for every $q \in P$.

We also obtain results which allow to decide in many cases when two norms are comparable. We have also localization results on the multiplication operator. Now we present the notation we use.

Notation. In the paper $k \geqslant 1$ denotes a fixed natural number; obviously $W^{0,p}(\Delta,\mu)=L^p(\Delta,\mu)$. All the measures we consider are Borel and positive on $\mathbf{R}$; if a measure is defined on a proper subset $E\subset\mathbf{R}$, we define it on $\mathbf{R}\backslash E$ as the zero measure. Also, all the weights are non-negative Borel measurable functions defined on $\mathbf{R}$. If the measure does not appear explicitly, we mean that we are using Lebesgue measure. We always work with measures which satisfy the decomposition $d\mu_j = d(\mu_j)_s + d(\mu_j)_{ac} = d(\mu_j)_s + h\,dx$, where $(\mu_j)_s$ is singular with respect to Lebesgue measure, $(\mu_j)_{ac}$ is absolutely continuous with respect to Lebesgue measure and h is a Lebesgue measurable function (which can be infinite in a set of positive Lebesgue measure); obviously the Radon–Nikodym Theorem gives that every σ-finite measure belongs to this class. Given a vectorial measure μ and a closed set E, we denote by $W^{k,p}(E,\mu)$ the space $W^{k,p}(\Delta\cap E, \mu|_E)$. We denote by supp v the support of the measure v. If A is a Borel set, $|A|$, χ_A, $\overline{A}$, $\operatorname{int}(A)$ and $\#A$ denote, respectively, the Lebesgue measure, the characteristic function, the closure, the interior and the cardinality of A. By $f^{(j)}$ we mean the jth distributional derivative of f. P and P_n denote respectively the set of polynomials and the set of polynomials with degree less than or equal to n. We say that an n-dimensional vector satisfies a one-dimensional property if each coordinate satisfies this property. Finally, the constants in the formulae can vary from line to line and even in the same line.

The outline of the paper is as follows. Section 2 presents most of the definitions we need to state our results. We prove some useful results on Sobolev spaces in Section 3. Section 4 is dedicated to the proof of the results for the multiplication operator.

2. DEFINITIONS AND RESULTS

Obviously one of our main problems is to define correctly the space $W^{k,p}(\Delta, \mu)$. There are two natural definitions:

(1) $W^{k,p}(\Delta, \mu)$ is the biggest space of (classes of) functions f which are regular enough to have $\|f\|_{W^{k,p}(\Delta,\mu)} < \infty$.

(2) $W^{k,p}(\Delta, \mu)$ is the closure of a good set of functions (e.g., $C^\infty(\mathbf{R})$ or P) with the norm $\|\cdot\|_{W^{k,p}(\Delta,\mu)}$.

However, both approaches have serious difficulties:

We consider first the approach (1). It is clear that the derivatives $f^{(j)}$ must be distributional derivatives in order to have a complete Sobolev space. Therefore we need to restrict the measures μ to a class of p-admissible measures (see Definition 8). Roughly speaking μ is p-admissible if $(\mu_j)_s$, for $0 < j \leqslant k$, is concentrated on the set of points where $f^{(j)}$ is continuous, for every function f of the space, because otherwise $f^{(j)}$ is determined, up to zero-Lebesgue measure sets (see Definitions 4 and 9 below). This will force $(\mu_k)_s$ to be identically zero. However, there will be no restriction on the support of $(\mu_0)_s$.

This reasonable approach excludes norms appearing in the theory of Sobolev orthogonal polynomials. Even if we work with the simpler case of the weighted Sobolev spaces $W^{k,p}(\Delta, w)$ (measures without singular part) we must impose that w_j belongs to the class B_p (see Definition 2 below) in order to have a complete weighted Sobolev space (see [KO, RARP1]).

The approach (2) is simpler: we know that the completion of every normed space exists (e.g., $(C^\infty(\mathbf{R}), \|\cdot\|_{W^{k,p}(\Delta,\mu)})$ or $(P, \|\cdot\|_{W^{k,p}(\Delta,\mu)})$). We have two difficulties. The first one is evident: we do not have an explicit description of the Sobolev functions as in (1) (in Section 4 of [RARP2] there are several theorems which show that both definitions of Sobolev space are the same for p-admissible measures). The second problem is worse: The completion of a normed space is by definition a set of equivalence classes of Cauchy sequences. In many cases this completion is not a function space (see Theorem 3.1 below and its Remark).

However, since we need to work with the multiplication operator in $P^{k,p}(\Delta, \mu)$, we have to choose this second approach if μ is not p-admissible.

First of all, we explain the definition of generalized Sobolev space in [RARP1]. We start with some preliminary definitions.

Definition 1. We say that two positive functions u, v are comparable on the set A if there are positive constants c_1, c_2 such that $c_1 v(x) \leqslant u(x) \leqslant c_2 v(x)$ for almost every $x \in A$. Since measures and norms are functions on measurable sets and vectors, respectively, we can talk about comparable measures and comparable norms. We say that two vectorial weights or vectorial measures are comparable if each component is comparable.

In what follows, the symbol $a \asymp b$ means that a and b are comparable for a and b functions, measures or norms.

Obviously, the spaces $L^p(A, \mu)$ and $L^p(A, \nu)$ are the same and have comparable norms if μ and ν are comparable on A. Therefore, in order to obtain our results we can change a measure μ to any comparable measure ν.

Next, we shall define a class of weights which plays an important role in our results.

Definition 2. If $1 \leqslant p < \infty$, we say that a weight w belongs to $B_p([a, b])$ if and only if

$$w^{-1} \in L^{1/(p-1)}([a, b]).$$

Also, if J is any interval we say that $w \in B_p(J)$ if $w \in B_p(I)$ for every compact interval $I \subseteq J$. We say that a weight belongs to $B_p(J)$, where J is a union of disjoint intervals $\bigcup_{i \in A} J_i$, if it belongs to $B_p(J_i)$, for $i \in A$.

Observe that if $v \geqslant w$ in J and $w \in B_p(J)$, then $v \in B_p(J)$.

The class $B_p(\mathbf{R})$ contains the classical $A_p(\mathbf{R})$ weights appearing in harmonic analysis (see [Mu1, GR]). The classes $B_p(\Omega)$, with $\Omega \subseteq \mathbf{R}^n$, and $A_p(\mathbf{R}^n)$ $(1 < p < \infty)$ have been used in other definitions of weighted Sobolev spaces in [KO, K], respectively.

Definition 3. We denote by $AC([a, b])$ the set of functions absolutely continuous on $[a, b]$, i.e., the functions $f \in C([a, b])$ such that $f(x) - f(a) = \int_a^x f'(t)\, dt$ for all $x \in [a, b]$. If J is any interval, $AC_{loc}(J)$ denotes the set of functions absolutely continuous on every compact subinterval of J.

Definition 4. Let us consider $1 \leqslant p < \infty$ and a vectorial measure $\mu = (\mu_0, ..., \mu_k)$ with absolutely continuous part $w = (w_0, ..., w_k)$. For $0 \leqslant j \leqslant k$ we define the open set

$$\Omega_j := \{x \in \mathbf{R} : \exists \text{ an open neighbourhood } V \text{ of } x \text{ with } w_j \in B_p(V)\}.$$

Observe that we always have $w_j \in B_p(\Omega_j)$ for any $1 \leqslant p < \infty$ and $0 \leqslant j \leqslant k$. In fact, Ω_j is the greatest open set U with $w_j \in B_p(U)$. Obviously, Ω_j depends on w and p, although p and μ do not appear explicitly in the symbol Ω_j. Applying Hölder's inequality it is easy to check that if $f^{(j)} \in L^p(\Omega_j, w_j)$ with $1 \leqslant j \leqslant k$, then $f^{(j)} \in L^1_{loc}(\Omega_j)$ and $f^{(j-1)} \in AC_{loc}(\Omega_j)$.

Hypothesis. *From now on we assume that w_j is identically 0 on the complement of Ω_j.*

Remark. We need this hypothesis in order to have complete Sobolev spaces (see [KO, RARP1]). This hypothesis is satisfied, for example, if we can modify w_j in a set of zero Lebesgue measure in such a way that there exists a sequence $\alpha_n \searrow 0$ with $w_j^{-1}\{(\alpha_n, \infty]\}$ open for every n. If w_j is lower semicontinuous, then this condition is satisfied.

The following definitions also depend on w and p, although w and p do not appear explicitly.

Let us consider $1 \leqslant p < \infty$, $\mu = (\mu_0, ..., \mu_k)$ a vectorial measure and $y \in \Delta$. To obtain a greater regularity of the functions in a Sobolev space we construct a modification of the measure μ in a neighbourhood of y, using the following Muckenhoupt weighted version of Hardy's inequality (see [Mu2; M, p. 44]). This modified measure is equivalent in some sense to the original one (see Theorem A below).

Muckenhoupt inequality. Let us consider $1 \leqslant p < \infty$ and μ_0, μ_1 measures in $(a, b\,]$ with $w_1 := d\mu_1/dx$. Then there exists a positive constant c such that

$$\left\| \int_x^b g(t)\, dt \right\|_{L^p((a,b],\, \mu_0)} \leqslant c\, \|g\|_{L^p((a,b],\, \mu_1)}$$

for any measurable function g in $(a, b]$, if and only if

$$\sup_{a<r<b} \mu_0((a, r])\, \|w_1^{-1}\|_{L^{1/(p-1)}([r,b])} < \infty.$$

DEFINITION 5. A vectorial measure $\bar{\mu} = (\bar{\mu}_0, ..., \bar{\mu}_k)$ is a right completion of a vectorial measure $\mu = (\mu_0, ..., \mu_k)$ with respect to y, if $\bar{\mu}_k := \mu_k$ and there is an $\varepsilon > 0$ such that $\bar{\mu}_j := \mu_j$ on the complement of $(y, y+\varepsilon]$ and

$$\bar{\mu}_j := \mu_j + \tilde{\mu}_j, \qquad \text{on} \quad (y, y+\varepsilon] \qquad \text{for} \quad 0 \leqslant j < k,$$

where $\tilde{\mu}_j$ is any measure satisfying:

(i) $\tilde{\mu}_j((y, y+\varepsilon]) < \infty$,

(ii) $\Lambda_p(\tilde{\mu}_j, \bar{\mu}_{j+1}) < \infty$, with

$$\Lambda_p(v, \sigma) := \sup_{y<r<y+\varepsilon} v((y, r]) \left\| \left(\frac{d\sigma}{dx}\right)^{-1} \right\|_{L^{1/(p-1)}([r,\, y+\varepsilon])}.$$

The Muckenhoupt inequality guarantees that if $f^{(j)} \in L^p(\mu_j)$ and $f^{(j+1)} \in L^p(\bar{\mu}_{j+1})$, then $f^{(j)} \in L^p(\bar{\mu}_j)$. If we work with absolutely continuous measures, we also say that a vectorial weight $\bar{w}$ is a completion of μ (or of w).

EXAMPLE. It can be shown that the following construction is always a completion: we choose $\tilde{w}_j := 0$ if $\bar{w}_{j+1} \notin B_p((y, y+\varepsilon])$; if $\bar{w}_{j+1} \in B_p([y, y+\varepsilon])$ we set $\tilde{w}_j(x) := 1$ in $[y, y+\varepsilon]$; and if $\bar{w}_{j+1} \in B_p((y, y+\varepsilon]) \setminus B_p([y, y+\varepsilon])$ we take $\tilde{w}_j(x) := 1$ for $x \in [y+\varepsilon/2, y+\varepsilon]$, and

$$\tilde{w}_j(x) := \frac{d}{dx} \left\{ \left(\int_x^{y+\varepsilon} \bar{w}_{j+1}^{-1/(p-1)} \right)^{-p+1} \right\}$$

$$= \frac{(p-1)\, \bar{w}_{j+1}(x)^{-1/(p-1)}}{\left(\int_x^{y+\varepsilon} \bar{w}_{j+1}^{-1/(p-1)}\right)^p} \qquad \text{if} \quad 1 < p < \infty,$$

$$\tilde{w}_j(x) := \|\bar{w}_{j+1}^{-1}\|_{L^\infty([x, y+\varepsilon])}^{-1} + \frac{d}{dx} \left(\|\bar{w}_{j+1}^{-1}\|_{L^\infty([x, y+\varepsilon])}^{-1} \right) \qquad \text{if} \quad p = 1,$$

for $x \in (y, y+\varepsilon/2)$.

Remarks. (1) We can define a left completion of μ with respect to y in a similar way.

(2) If $\bar{w}_{j+1} \in B_p([y, y+\varepsilon])$, then $\Lambda_p(\tilde{\mu}_j, \bar{w}_{j+1}) < \infty$ for any measure $\tilde{\mu}_j$ with $\tilde{\mu}_j((y, y+\varepsilon]) < \infty$. In particular, $\Lambda_p(1, \bar{w}_{j+1}) < \infty$.

(3) If μ, v are comparable measures, $\bar{v}$ is a right completion of v if and only if it is comparable to a right completion $\bar{\mu}$ of μ.

(4) If μ, v are two vectorial measures with the same absolutely continuous part, then $\bar{\mu}$ is a right completion of μ if and only if it is a right completion of v.

DEFINITION 6. For $1 \leqslant p < \infty$ and a vectorial measure μ, we say that a point $y \in \mathbf{R}$ is right j-regular (respectively, left j-regular), if there exist $\varepsilon > 0$, a right completion $\bar{w}$ (respectively, left completion) of μ and $j < i \leqslant k$ such that $\bar{w}_i \in B_p([y, y+\varepsilon])$ (respectively, $B_p([y-\varepsilon, y])$). Also, we say that a point $y \in \mathbf{R}$ is j-regular, if it is right and left j-regular.

Remarks. (1) A point $y \in \mathbf{R}$ is right j-regular (respectively, left j-regular), if at least one of the following properties is satisfied:

(a) There exist $\varepsilon > 0$ and $j < i \leqslant k$ such that $w_i \in B_p([y, y+\varepsilon])$ (respectively, $B_p([y-\varepsilon, y])$). Here we have chosen $\tilde{w}_j = 0$.

(b) There exist $\varepsilon > 0$, $j < i \leqslant k$, $\alpha > 0$, and $\delta < (i-j)\,p - 1$, such that

$$w_i(x) \geqslant \alpha\, |x-y|^\delta, \qquad \text{for almost every} \quad x \in [y, y+\varepsilon]$$

(respectively, $[y-\varepsilon, y]$). See Lemma 3.4 in [RARP1].

(2) If y is right j-regular (respectively, left), then it is also right i-regular (respectively, left) for each $0 \leqslant i \leqslant j$.

(3) We can take $i=j+1$ in this definition since by the second remark after Definition 5 we can choose $\bar{w}_l = w_l + 1 \in B_p([y, y+\varepsilon])$ for $j<l<i$, if $j+1<i$.

(4) If μ, ν are two vectorial measures with the same absolutely continuous part, then y is right j-regular (respectively, left) with respect to μ if and only if it is right j-regular (respectively, left) with respect to ν.

When we use this definition we think of a point $\{b\}$ as the union of two half-points $\{b^+\}$ and $\{b^-\}$. With this convention, each one of the following sets

$$(a, b) \cup (b, c) \cup \{b^+\} = (a, b) \cup [b^+, c) \neq (a, c),$$

$$(a, b) \cup (b, c) \cup \{b^-\} = (a, b^-] \cup (b, c) \neq (a, c),$$

has two connected components, and the set

$$(a, b) \cup (b, c) \cup \{b^-\} \cup \{b^+\} = (a, b) \cup (b, c) \cup \{b\} = (a, c)$$

is connected.

We only use this convention in order to study the sets of continuity of functions: we want that if $f \in C(A)$ and $f \in C(B)$, where A and B are union of intervals, then $f \in C(A \cup B)$. With the usual definition of continuity in an interval, if $f \in C([a, b)) \cap C([b, c])$ then we do not have $f \in C([a, c])$. Of course, we have $f \in C([a, c])$ if and only if $f \in C([a, b^-]) \cap C([b^+, c])$, where, by definition, $C([b^+, c]) = C([b, c])$ and $C([a, b^-]) = C([a, b])$. This idea can be formalized with a suitable topological space.

Let us introduce some notation. We denote by $\Omega^{(j)}$ the set of j-regular points or half-points, i.e., $y \in \Omega^{(j)}$ if and only if y is j-regular, we say that $y^+ \in \Omega^{(j)}$ if and only if y is right j-regular, and we say that $y^- \in \Omega^{(j)}$ if and only if y is left j-regular. Obviously, $\Omega^{(k)} = \emptyset$ and $\Omega_{j+1} \cup \cdots \cup \Omega_k \subseteq \Omega^{(j)}$. Observe that $\Omega^{(j)}$ depends on p (see Definition 6).

Remark. If $0 \leqslant j < k$ and I is an interval, $I \subseteq \Omega^{(j)}$, then the set $I \backslash (\Omega_{j+1} \cup \cdots \cup \Omega_k)$ is discrete (see the Remark before Definition 7 in [RARP1]).

DEFINITION 7. We say that a function h belongs to the class $AC_{loc}(\Omega^{(j)})$ if $h \in AC_{loc}(I)$ for every connected component I of $\Omega^{(j)}$.

DEFINITION 8. We say that the vectorial measure $\mu = (\mu_0, ..., \mu_k)$ is p-admissible if $(\mu_j)_s(\mathbf{R} \backslash \Omega^{(j)}) = 0$ for $1 \leqslant j \leqslant k$.

We use the letter p in p-admissible in order to emphasize the dependence on p (recall that $\Omega^{(j)}$ depends on p).

Remarks. (1) There is no condition on supp $(\mu_0)_s$.

(2) We have $(\mu_k)_s \equiv 0$, since $\Omega^{(k)} = \varnothing$.

(3) Every absolutely continuous measure is p-admissible.

DEFINITION 9 (Sobolev Space). Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ a p-admissible vectorial measure. We define the Sobolev space $W^{k,p}(\Delta, \mu)$ as the space of equivalence classes of

$$V^{k,p}(\Delta, \mu) := \{ f\colon \Delta \to \mathbf{C} / f^{(j)} \in AC_{loc}(\Omega^{(j)}) \text{ for } 0 \leqslant j < k \text{ and } \\ \| f^{(j)} \|_{L^p(\Delta, \mu_j)} < \infty \text{ for } 0 \leqslant j \leqslant k \},$$

with respect to the seminorm

$$\| f \|_{W^{k,p}(\Delta, \mu)} := \left(\sum_{j=0}^{k} \| f^{(j)} \|^p_{L^p(\Delta, \mu_j)} \right)^{1/p}.$$

Remarks. This definition is natural since when the $(\mu_j)_s$-measure of the set where $|f^{(j)}|$ is not continuous is positive, the integral $\int |f^{(j)}|^p \, d(\mu_j)_s$ does not make sense. If we consider Sobolev spaces with real valued functions every result in this paper also holds.

At this moment we can consider also norms like the following:

$$\| f \|^p = \int_{-1}^{1} |f|^p + \int_{-1}^{0} |x|^{p-1} \, |f'|^p + \int_{0}^{1} |f'|^p + |f(0^+)|^p ,$$

$$\| f \|^p = \int_{0}^{1} |f|^p + \int_{0}^{1} |f'|^p + |f(0^+)|^p .$$

In the second example, we can write $|f(0)|^p$ instead of $|f(0^+)|^p$, since f is not defined at the left of 0, and then this causes no confusion. Obviously we always write $(a+b)\, \delta_{x_0}$ instead of $a\, \delta_{x_0^-} + b\, \delta_{x_0^+}$.

DEFINITION 10. Let us consider $1 \leqslant p < \infty$ and μ a p-admissible vectorial measure. Let us define the space $\mathscr{K}(\Delta, \mu)$ as

$$\mathscr{K}(\Delta, \mu) := \{ g\colon \Omega^{(0)} \to \mathbf{C} / g \in V^{k,p}(\overline{\Omega^{(0)}}, \mu|_{\Omega^{(0)}}),\ \| g \|_{W^{k,p}(\overline{\Omega^{(0)}}, \mu|_{\Omega^{(0)}})} = 0 \}.$$

$\mathscr{K}(\Delta, \mu)$ is the equivalence class of 0 in $W^{k,p}(\overline{\Omega^{(0)}}, \mu|_{\Omega^{(0)}})$. It plays an important role in the general theory of Sobolev spaces and in the study of the multiplication operator in Sobolev spaces in particular (see [RARP1, RARP2], Theorem A below, and Theorem C in Section 4).

DEFINITION 11. Let us consider $1 \leqslant p < \infty$ and μ a p-admissible vectorial measure. We say that (Δ, μ) belongs to the class $\mathscr{C}_0$ if there exist compact sets M_n, which are a finite union of compact intervals, such that

(i) M_n intersects at most a finite number of connected components of $\Omega_1 \cup \cdots \cup \Omega_k$,

(ii) $\mathscr{K}(M_n, \mu) = \{0\}$,

(iii) $M_n \subseteq M_{n+1}$,

(iv) $\bigcup_n M_n = \Omega^{(0)}$.

We say that (Δ, μ) belongs to the class $\mathscr{C}$ if there exists a measure $\mu'_0 = \mu_0 + \sum_{m \in D} c_m \delta_{x_m}$ with $c_m > 0$, $\{x_m\} \subset \Omega^{(0)}$, $D \subseteq \mathbf{N}$ and $(\Delta, \mu') \in \mathscr{C}_0$, where $\mu' = (\mu'_0, \mu_1, ..., \mu_k)$ is minimal in the following sense: there exists $\{M_n\}$ corresponding to $(\Delta, \mu') \in \mathscr{C}_0$ such that if $\mu''_0 = \mu'_0 - c_{m_0} \delta_{x_{m0}}$ with $m_0 \in D$ and $\mu'' = (\mu''_0, \mu_1, ..., \mu_k)$, then $\mathscr{K}(M_n, \mu'') \neq \{0\}$ if $x_{m_0} \in M_n$.

Remarks. (1) The condition $(\Delta, \mu) \in \mathscr{C}$ is not very restrictive. In fact, the proof of Theorem A below (see [RARP1, Theorem 4.3]) gives that if $\Omega^{(0)} \setminus (\Omega_1 \cup \cdots \cup \Omega_k)$ has only a finite number of points in each connected component of $\Omega^{(0)}$, then $(\bar{\Omega}, \mu) \in \mathscr{C}$. If furthermore $\mathscr{K}(\Delta, \mu) = \{0\}$, we have $(\Delta, \mu) \in \mathscr{C}_0$.

(2) Since the restriction of a function of $\mathscr{K}(\Delta, \mu)$ to M_n is in $\mathscr{K}(M_n, \mu)$ for every n, then $(\Delta, \mu) \in \mathscr{C}_0$ implies $\mathscr{K}(\Delta, \mu) = \{0\}$.

(3) If $(\Delta, \mu) \in \mathscr{C}_0$, then $(\Delta, \mu) \in \mathscr{C}$, with $\mu' = \mu$.

(4) The proof of Theorem A below gives that if for every connected component Λ of $\Omega_1 \cup \cdots \cup \Omega_k$ we have $\mathscr{K}(\bar{\Lambda}, \mu) = \{0\}$, then $(\Delta, \mu) \in \mathscr{C}_0$. Condition $\# \operatorname{supp} \mu_0 |_{\bar{\Lambda} \cap \Omega^{(0)}} \geqslant k$ implies $\mathscr{K}(\bar{\Lambda}, \mu) = \{0\}$.

The next results, proved in [RARP1], play a central role in the theory of Sobolev spaces with respect to measures (see the proofs in [RARP1, Theorems 4.3 and 5.1]).

THEOREM A. *Let us suppose that $1 \leqslant p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ is a p-admissible vectorial measure. Let K_j be a finite union of compact intervals contained in $\Omega^{(j)}$, for $0 \leqslant j < k$ and $\bar{\mu}$ a right (or left) completion of μ. Then:*

(a) *If $(\Delta, \mu) \in \mathscr{C}_0$ there exist positive constants $c_1 = c_1(K_0, ..., K_{k-1})$ and $c_2 = c_2(\bar{\mu}, K_0, ..., K_{k-1})$ such that*

$$c_1 \sum_{j=0}^{k-1} \|g^{(j)}\|_{L^\infty(K_j)} \leqslant \|g\|_{W^{k,p}(\Delta, \mu)},$$

$$c_2 \|g\|_{W^{k,p}(\Delta, \bar{\mu})} \leqslant \|g\|_{W^{k,p}(\Delta, \mu)}, \qquad \forall g \in V^{k,p}(\Delta, \mu).$$

(b) *If* $(\Delta, \mu) \in \mathscr{C}$ *there exist positive constants* $c_3 = c_3(K_0, ..., K_{k-1})$ *and* $c_4 = c_4(\bar{\mu}, K_0, ..., K_{k-1})$ *such that for every* $g \in V^{k,p}(\Delta, \mu)$, *there exists* $g_0 \in V^{k,p}(\Delta, \mu)$, *independent of* $K_0, ..., K_{k-1}$, c_3, c_4 *and* $\bar{\mu}$, *with*

$$\|g_0 - g\|_{W^{k,p}(\Delta,\mu)} = 0,$$

$$c_3 \sum_{j=0}^{k-1} \|g_0^{(j)}\|_{L^\infty(K_j)} \leqslant \|g_0\|_{W^{k,p}(\Delta,\mu)} = \|g\|_{W^{k,p}(\Delta,\mu)},$$

$$c_4 \|g_0\|_{W^{k,p}(\Delta,\bar{\mu})} \leqslant \|g\|_{W^{k,p}(\Delta,\mu)}.$$

Furthermore, if g_0, f_0 *are these representatives of* g, f *respectively, we have for the same constants* c_3, c_4

$$c_3 \sum_{j=0}^{k-1} \|g_0^{(j)} - f_0^{(j)}\|_{L^\infty(K_j)} \leqslant \|g - f\|_{W^{k,p}(\Delta,\mu)},$$

$$c_4 \|g_0 - f_0\|_{W^{k,p}(\Delta,\bar{\mu})} \leqslant \|g - f\|_{W^{k,p}(\Delta,\mu)}.$$

Remark. Theorem A is proved in [RARP1] with the additional hypothesis that $\tilde{\mu} := \bar{\mu} - \mu$ is absolutely continuous, since [RARP1] only uses absolutely continuous completions, but the same proof also works in the general case.

THEOREM B. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, ..., \mu_k)$ *a p-admissible vectorial measure with* $(\Delta, \mu) \in \mathscr{C}$. *Then the Sobolev space* $W^{k,p}(\Delta, \mu)$ *is complete.*

3. RESULTS ON SOBOLEV SPACES

We start this section with a technical result which shows how to modify a measure in order to have $(\Delta, \mu) \in \mathscr{C}_0$. We use this proposition in the proof of Corollary 3.2 below.

PROPOSITION 3.1. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, ..., \mu_k)$ *a p-admissible vectorial measure. Then there exists a measure* $\mu_0^* \geqslant \mu_0$ *with* $\mu_0^* - \mu_0$ *discrete and finite,* $(\mu_0^* - \mu_0)(\mathbf{R} \setminus \Omega^{(0)}) = 0$, *and such that* $\mu^* := (\mu_0^*, \mu_1, ..., \mu_k)$ *is p-admissible and* $(\Delta, \mu^*) \in \mathscr{C}_0$. *We have also* $V^{k,p}(\Delta, \mu) \cap L^\infty(\Omega^{(0)}) \subseteq V^{k,p}(\Delta, \mu^*)$ *and*

$$\|f\|_{W^{k,p}(\Delta,\mu^*)} \leqslant \|f\|_{W^{k,p}(\Delta,\mu)} + \|f\|_{L^\infty(\Omega^{(0)})},$$

for every $f \in V^{k,p}(\Delta, \mu)$.

Proof. Let us consider the connected components $\{A_m\}_{m=1}^{M}$ ($M \in \mathbf{N} \cup \{\infty\}$) of $\Omega_1 \cup \cdots \cup \Omega_k$. For each m, choose k points $x_m^1, ..., x_m^k \in A_m$, and now define the measure

$$\mu_0^* := \mu_0 + \frac{1}{k} \sum_{i=1}^{k} \sum_{m=1}^{M} 2^{-m} \delta_{x_m^i}.$$

Obviously $\mu_0^* - \mu_0$ is discrete and finite, and $(\mu_0^* - \mu_0)(\mathbf{R} \backslash \Omega^{(0)}) = 0$. Obviously μ^* is p-admissible since μ is p-admissible. We see now that $\mathscr{K}(\bar{A}_m, \mu^*) = \{0\}$.

Let us consider $q \in \mathscr{K}(\bar{A}_m, \mu^*)$. For each $y \in A_m$, there is a $1 \leqslant j \leqslant k$ with $y \in \Omega_j$. Let I be the connected component of Ω_j which contains the point y. If w_j denotes the absolutely continuous part of μ_j, we have that

$$\int_I |q^{(j)}(x)|^p \, w_j(x) \, dx = 0,$$

since $q \in \mathscr{K}(\bar{A}_m, \mu^*)$. Hölder's inequality gives

$$\|q^{(j)}\|_{L^1(I')} \leqslant \|q^{(j)}\|_{L^p(I', w_j)} \|w_j^{-1}\|_{L^{1/(p-1)}(I')} = 0,$$

for every compact interval $I' \subset I$, since $w_j \in B_p(\Omega_j)$. Then $\|q^{(j)}\|_{L^1(I)} = 0$ and since $q^{(j-1)}$ is locally absolutely continuous in I, it has to be constant in I, and consequently $q^{(j)} \equiv 0$ in I. We have that $q|_I \in P_{j-1} \subseteq P_{k-1}$. Then we obtain $q|_{A_m} \in P_{k-1}$, since A_m is a connected set. We conclude $q = 0$ in A_m since $q(x_m^1) = \cdots = q(x_m^k) = 0$. The same argument gives $\mathscr{K}(J, \mu) = \{0\}$ for every closed interval $J \subset \bar{A}_m$ with $x_m^1, ..., x_m^k \in J$.

For each m and n, let us consider a compact interval $J_{n,m}$ with $x_m^1, ..., x_m^k \in J_{n,m}$, $J_{n,m} \subseteq J_{n+1,m}$ and $\bigcup_n J_{n,m} = \bar{A}_m \cap \Omega^{(0)}$. We define now $M_n := \bigcup_{m \in D_n} J_{n,m}$, where $D_n := \{m : |A_m| \geqslant 1/n$ and $A_m \cap (-n, n) \neq \emptyset\}$. Since $\# D_n \leqslant 2n^2 + 1$ and $\mathscr{K}(J_{n,m}, \mu^*) = \{0\}$, this choice of $\{M_n\}$ gives $(\Delta, \mu^*) \in \mathscr{C}_0$.

Assume now that $f \in V^{k,p}(\Delta, \mu) \cap L^\infty(\Omega^{(0)})$. We have that

$$|f(x_m^i)| \leqslant \|f\|_{L^\infty(\Omega^{(0)})},$$

for every m and i, since f is continuous at x_m^i. We have also

$$\int |f|^p \, d\mu_0^* = \int |f|^p \, d\mu_0 + \int |f|^p \, d(\mu_0^* - \mu_0) \leqslant \|f\|_{L^p(\mu_0)}^p + \|f\|_{L^\infty(\Omega^{(0)})}^p,$$

$$\|f\|_{L^p(\mu_0^*)} \leqslant \|f\|_{L^p(\mu_0)} + \|f\|_{L^\infty(\Omega^{(0)})},$$

$$\|f\|_{W^{k,p}(\Delta, \mu^*)} \leqslant \|f\|_{W^{k,p}(\Delta, \mu)} + \|f\|_{L^\infty(\Omega^{(0)})},$$

since $(\mu_0^* - \mu_0)(\mathbf{R}) = (\mu_0^* - \mu_0)(\Omega^{(0)}) \leqslant 1$. Then we have $V^{k,p}(\Delta, \mu) \cap L^\infty(\Omega^{(0)}) \subseteq V^{k,p}(\Delta, \mu^*)$. ∎

An immediate computation gives the following technical result.

LEMMA 3.1. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, ..., \mu_k)$ *a vectorial measure with*

$$d\mu_{j+1}(x) \leqslant c_1^p \, |x - x_0|^p \, d\mu_j(x),$$

for $0 \leqslant j < k$, $x_0 \in \mathbf{R}$ *and* x *in an interval* I. *Let* $\varphi \in C^k(\mathbf{R})$ *be such that* $\operatorname{supp} \varphi' \subseteq [\lambda_1, \lambda_1 + t] \cup [\lambda_2, \lambda_2 + t] \subseteq I$, *with* $\lambda_1 + t < \lambda_2$, $\max\{|\lambda_1 - x_0|, |\lambda_1 + t - x_0|, |\lambda_2 - x_0|, |\lambda_2 + t - x_0|\} \leqslant c_2 t$ *and* $\|\varphi^{(j)}\|_{L^\infty(I)} \leqslant c_3 t^{-j}$ *for* $0 \leqslant j \leqslant k$. *Then, there is a positive constant* c_0 *which is independent of* I, x_0, λ_1, λ_2, t, μ, φ, *and* g *such that*

$$\|\varphi g\|_{W^{k,p}(\Delta, \mu)} \leqslant c_0 \|g\|_{W^{k,p}(I, \mu)},$$

for every $g \in C^k(\mathbf{R})$ *with* $\operatorname{supp}(\varphi g) \subseteq I$.

Remarks. (1) The constant c_0 can depend on c_1, c_2, c_3, p, and k.

(2) In the proof we only use the hypothesis $g \in C^k(\mathbf{R})$ to assure that $\int |g^{(j)}|^p \, d\mu_j$ has sense (although it can be infinite). Therefore, if μ is p-admissible, the result is also true for every $g \in V^{k,p}(\Delta, \mu)$ with $\operatorname{supp}(\varphi g) \subseteq I$.

(3) Condition $d\mu_{j+1}(x) \leqslant c_1^p \, |x - x_0|^p \, d\mu_j(x)$ means that μ_{j+1} is absolutely continuous with respect to μ_j, and that the Radon–Nikodym derivative satisfies $d\mu_{j+1}/d\mu_j \leqslant c_1^p \, |x - x_0|^p$. Proposition 3.2 below shows that this condition is not as restrictive as it seems, since many weights with analytic singularities can be modified in order to satisfy it.

We define now the functions

$$\log_1 x := -\log x, \ \log_2 x := \log(\log_1 x), ..., \log_n x := \log(\log_{n-1} x).$$

With this definition we have the following result, which is a consequence of Muckenhoupt inequality.

PROPOSITION 3.2. *Let us consider* $1 \leqslant p < \infty$ *and* $w = (w_0, ..., w_k)$ *a finite vectorial weight in* (a, b). *Assume also that there exist* $0 \leqslant k_0 < k$, $x_0 \in \mathbf{R}$, *a neighbourhood* U *of* x_0, $n \in \mathbf{N}$, $c_i > 0$, $\varepsilon_i \geqslant 0$ *and* $\alpha_i, \gamma_1^i, ..., \gamma_n^i \in \mathbf{R}$ *for* $k_0 \leqslant i \leqslant k$ *such that*

(i) $w_i(x) \asymp e^{-c_i |x - x_0|^{-\varepsilon_i}} \, |x - x_0|^{\alpha_i} \log_1^{\gamma_1^i} |x - x_0| \cdots \log_n^{\gamma_n^i} |x - x_0|$ *for* $x \in U$ *and* $k_0 \leqslant i \leqslant k$,

(ii) $(1+\alpha_i)/p \notin \mathbf{N}$ *if* $\varepsilon_i = 0$ *and* $k_0 < i \leqslant k$,

(iii) $w_k \notin B_p(U)$.

Then there exists a weight w^* *in* (a, b) *such that the Sobolev norms* $W^{k,p}([a,b], w)$ *and* $W^{k,p}([a,b], w^*)$ *are comparable for every function in* $W^{k,p}([a,b], w)$ *and satisfying*

$$w^*_{j+1}(x) \leqslant c\,|x-x_0|^p\, w^*_j(x)\,,$$

for $k'_0 \leqslant j < k$ *and* $x \in U$, *for some* $k_0 \leqslant k'_0 < k$. *Furthermore, if* $k_0 \neq k'_0$ *then we have* $w^*_{k'_0} \in B_p(U)$.

The following result reveals a big problem when dealing with the completion of P. Furthermore, it allows to prove Theorem 4.5 about the multiplication operator.

THEOREM 3.1. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, ..., \mu_k)$ *a finite vectorial measure with* Δ *a compact set. Assume that there exist* $x_0 \in \mathbf{R}$ *and* $0 \leqslant k_0 \leqslant k$ *with* $\mu_{k_0}(\{x_0\}) = 0$ *and satisfying the following property if* $k_0 < k$: *there exist an open neighbourhood* U *of* x_0 *and* $c > 0$ *such that*

$$d\mu_{j+1}(x) \leqslant c\,|x-x_0|^p\, d\mu_j(x)\,,$$

for $x \in U$ *and* $k_0 \leqslant j < k$. *Let us define*

$$v := (0, ..., 0, \alpha_{k_0}\delta_{x_0}, \alpha_{k_0+1}\delta_{x_0}, ..., \alpha_k\delta_{x_0})$$

and $N := \#\{k_0 \leqslant j \leqslant k : \alpha_j > 0\}$. *Given a Cauchy sequence* $\{q_n\} \subset P$ *in* $W^{k,p}(\Delta, \mu)$ *and* $u_{k_0}, ..., u_k \in \mathbf{R}$ *there exists a Cauchy sequence* $\{r_n\} \subset P$ *in* $W^{k,p}(\Delta, \mu)$ *with* $\lim_{n\to\infty} \|q_n - r_n\|_{W^{k,p}(\Delta,\mu)} = 0$ *and* $r_n^{(j)}(x_0) = u_j$ *for* $k_0 \leqslant j \leqslant k$. *Consequently* $P^{k,p}(\Delta, \mu + v)$ *is isomorphic to* $P^{k,p}(\Delta, \mu) \times \mathbf{R}^N$.

Remark. Observe that $P^{k,p}(\Delta, \mu+v)$ is not a space of functions even when $P^{k,p}(\Delta, \mu)$ is a space of functions. In fact, if $q \in P$ is an element of $P^{k,p}(\Delta, \mu)$, then it represents $\mathbf{R}^N$ elements of $P^{k,p}(\Delta, \mu+v)$, and therefore there are infinitely many equivalence classes in $P^{k,p}(\Delta, \mu+v)$ whose restriction to $P^{k,p}(\Delta, \mu)$ coincides with q. Hence, the values $f^{(j)}(x_0)$ for $k_0 \leqslant j \leqslant k$ do not represent anything related with the derivatives of $f \in P^{k,p}(\Delta, \mu+v)$.

Proof. It is enough to see that, given sequences $\{v^n_{k_0}\}, ..., \{v^n_k\} \subset \mathbf{R}$, there exists a sequence $\{s_n\} \subset P$ converging to 0 in the norm of $W^{k,p}(\Delta, \mu)$ with $s_n^{(j)}(x_0) = v^n_j$ for $k_0 \leqslant j \leqslant k$, since then we can take $r_n := q_n - s_n$ with $v^n_j := q_n^{(j)}(x_0) - u_j$.

Let us consider the polynomial $h_n \in P_{k-k_0}$ with $h_n^{(j-k_0)}(x_0) = v_j^n$ for $k_0 \leqslant j \leqslant k$, a function $\varphi \in C_c^\infty(\mathbf{R})$ with $0 \leqslant \varphi \leqslant 1$ and

$$\varphi(x) := \begin{cases} 1, & \text{if} \quad x \in [-1, 1], \\ 0, & \text{if} \quad x \notin [-2, 2], \end{cases}$$

and the functions

$$\varphi_t(x) := \varphi\left(\frac{x - x_0}{t}\right),$$

for each $0 < t \leqslant t_0$, where t_0 is any positive number with $\operatorname{supp} \varphi_{t_0} \subset U$. For each $n \in \mathbf{N}$, define the function $g_n := h_n \varphi_{t_n}$, where $\{t_n\}$ is a sequence converging to 0, with $0 < t_n < t_0$, which will be chosen later.

Let us define $f_n := g_n$ if $k_0 = 0$ and

$$f_n(x) := \int_{x_0 + 2t_n}^{x} g_n(t) \frac{(x-t)^{k_0 - 1}}{(k_0 - 1)!} \, dt,$$

otherwise. Since we have

$$f_n^{(j)}(x) = \int_{x_0 + 2t_n}^{x} g_n(t) \frac{(x-t)^{k_0 - j - 1}}{(k_0 - j - 1)!} \, dt,$$

for $0 \leqslant j < k_0$, μ is finite and Δ is compact, we obtain that

$$\|f_n^{(j)}\|_{L^p(\Delta, \mu_j)} \leqslant c \, \|f_n^{(j)}\|_{L^\infty(\Delta)} \leqslant c \, \|g_n\|_{L^1(\mathbf{R})} \leqslant c \, \|h_n\|_{L^1([x_0 - 2t_n, x_0 + 2t_n])}, \tag{3.1}$$

for $0 \leqslant j < k_0$. If $k_0 < k$, Lemma 3.1 gives that

$$\begin{aligned} \sum_{j=k_0}^{k} \|f_n^{(j)}\|_{L^p(\Delta, \mu_j)} &= \sum_{j=k_0}^{k} \|g_n^{(j-k_0)}\|_{L^p(\Delta, \mu_j)} \\ &\leqslant c \, \|h_n \varphi_{t_n}\|_{W^{k-k_0, p}([x_0 - 2t_n, x_0 + 2t_n], (\mu_{k0}, ..., \mu_k))} \\ &\leqslant c \, \|h_n\|_{W^{k-k_0, p}([x_0 - 2t_n, x_0 + 2t_n], (\mu_{k_0}, ..., \mu_k))}. \end{aligned} \tag{3.2}$$

We can apply Lemma 3.1 since

$$\operatorname{supp} \varphi'_{t_n} \subseteq [x_0 - 2t_n, x_0 - t_n] \cup [x_0 + t_n, x_0 + 2t_n] \subset [x_0 - 2t_n, x_0 + 2t_n],$$

$$\max\{|-2t_n|, |-t_n|, t_n, 2t_n\} = 2t_n,$$

$$\|\varphi_{t_n}^{(j)}\|_{L^\infty(\mathbf{R})} = t_n^{-j} \, \|\varphi^{(j)}\|_{L^\infty(\mathbf{R})} \leqslant c t_n^{-j} \qquad \text{for} \quad 0 \leqslant j \leqslant k - k_0,$$

$$\operatorname{supp} g_n = \operatorname{supp}(h_n \varphi_{t_n}) \subseteq [x_0 - 2t_n, x_0 + 2t_n].$$

If $k_0 = k$, inequality (3.2) is also true since

$$\|f_n^{(k)}\|_{L^p(\Delta, \mu_k)} = \|g_n\|_{L^p(\Delta, \mu_k)} \leqslant \|h_n\|_{L^p([x_0 - 2t_n, x_0 + 2t_n], \mu_k)}.$$

Inequalities (3.1) and (3.2) and the fact $\mu_{k_0}(\{x_0\}) = \cdots = \mu_k(\{x_0\}) = 0$ allow us to choose t_n small enough in order that

$$\|f_n\|_{W^{k,p}(\Delta, \mu)} < \frac{1}{n}. \tag{3.3}$$

If Δ^* is the convex hull of $\Delta \cup \{x_0\}$, we can choose $p_n \in P$ such that

$$\|f_n^{(j)} - p_n^{(j)}\|_{L^\infty(\Delta^*)} < \frac{1}{n}, \tag{3.4}$$

for $0 \leqslant j \leqslant k$, since $f_n \in C^\infty(\mathbf{R})$. This is deduced from the compactness of Δ and Bernstein's proof of the Weierstrass Theorem, where the Bernstein polynomials approximate any function in $C^k([a, b])$ uniformly up to the k-th derivative (see, e.g., [D, p. 113]).

In particular, we have that

$$|f_n^{(j)}(x_0) - p_n^{(j)}(x_0)| < \frac{1}{n},$$

for $0 \leqslant j \leqslant k$. If we consider the polynomial $\varepsilon_n \in P_k$ with

$$\varepsilon_n^{(j)}(x_0) = f_n^{(j)}(x_0) - p_n^{(j)}(x_0) \ ,$$

for $0 \leqslant j \leqslant k$, then there exists a positive constant c, which only depends on Δ^*, with

$$\|\varepsilon_n^{(j)}\|_{L^\infty(\Delta^*)} < \frac{c}{n}, \tag{3.5}$$

for $0 \leqslant j \leqslant k$. Therefore, the polynomial $s_n := p_n + \varepsilon_n$ satisfies

$$s_n^{(j)}(x_0) = p_n^{(j)}(x_0) + \varepsilon_n^{(j)}(x_0) = f_n^{(j)}(x_0) = g_n^{(j-k_0)}(x_0) = h_n^{(j-k_0)}(x_0) = v_j^n,$$

for $k_0 \leqslant j \leqslant k$, and (3.3), (3.4), and (3.5) show that there is a positive constant c, which does not depend on n, with

$$\|s_n\|_{W^{k,p}(\Delta, \mu)} < \frac{c}{n}.$$

This finishes the proof of Theorem 3.1. ∎

The proof of Theorem 3.1 gives the following result.

Corollary 3.1. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, ..., \mu_k)$ *a finite vectorial measure with* Δ *a compact set. Assume that there exist* $x_0 \in \mathbf{R}$ *and* $0 \leqslant k_0 \leqslant k$ *with* $\mu_{k_0}(\{x_0\}) = 0$ *and satisfying the following property if* $k_0 < k$: *there exist an open neighbourhood* U *of* x_0 *and* $c > 0$ *such that*

$$d\mu_{j+1}(x) \leqslant c\, |x - x_0|^p\, d\mu_j(x),$$

for $x \in U$ *and* $k_0 \leqslant j < k$. *Given sequences* $\{v_{k_0}^n\}, ..., \{v_k^n\} \subset \mathbf{R}$ *there exists a sequence* $\{s_n\} \subset P$ *converging to* 0 *in the norm of* $W^{k,p}(\Delta, \mu)$ *with* $s_n^{(j)}(x_0) = v_j^n$ *for* $k_0 \leqslant j \leqslant k$.

We have also the following consequences of Theorem 3.1.

Corollary 3.2. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, ..., \mu_k)$ *a finite vectorial measure with* Δ *a compact set. Assume that there exist* $x_0 \in \mathbf{R}$, $0 \leqslant k_0 < k$, *an open neighbourhood* U *of* x_0 *and* $c > 0$ *such that*

$$d\mu_{j+1}(x) \leqslant c\, |x - x_0|^p\, d\mu_j(x),$$

for $x \in U$ *and* $k_0 \leqslant j < k$. *Then* x_0 *is neither right nor left* k_0*-regular.*

Proof. Without loss of generality we can assume that μ is absolutely continuous, since the *j*-regularity just depends on the absolutely continuous part of the measure. Consequently μ is *p*-admissible. Assume that x_0 is right or left k_0-regular and consider the measure μ^* as in Proposition 3.1 with the additional condition $x_m^i \neq x_0$ for every m and i. Then $(\Delta, \mu^*) \in \mathscr{C}_0$ and we have by Theorem A

$$|g^{(k_0)}(x_0)| \leqslant c\, \|g\|_{W^{k,p}(\Delta, \mu^*)}, \qquad \forall g \in V^{k,p}(\Delta, \mu),$$

and consequently

$$|q^{(k_0)}(x_0)| \leqslant c\, \|q\|_{W^{k,p}(\Delta, \mu^*)}, \qquad \forall q \in P,$$

since μ^* finite and Δ compact imply $\mu^* \in \mathscr{M}$. The measure μ^* satisfies the hypotheses in Theorem 3.1 and therefore there exists a sequence of polynomials $\{r_n\}$ with $r_n^{(k_0)}(x_0) = 1$ and $\lim_{n \to \infty} \|r_n\|_{W^{k,p}(\Delta, \mu^*)} = 0$, which contradicts the last inequality.

Corollary 3.3. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, ..., \mu_k)$ *a finite vectorial measure with* Δ *a compact set. Assume that there exist* $x_0 \in \mathbf{R}$ *and* $0 \leqslant k_0 \leqslant k$ *with* $\mu_{k_0}(\{x_0\}) = 0$ *and satisfying the following property if* $k_0 < k$: *there exists an open neighbourhood* U *of* x_0 *such that* $\mu_j(U) = 0$ *for* $k_0 < j \leqslant k$. *Let us define*

$$v := (0, ..., 0, \alpha_{k_0}\delta_{x_0}, \alpha_{k_0+1}\delta_{x_0}, ..., \alpha_k\delta_{x_0})$$

and $N := \#\{k_0 \leqslant j \leqslant k : \alpha_j > 0\}$. *Given a Cauchy sequence* $\{q_n\} \subset P$ *in* $W^{k,p}(\Delta, \mu)$ *and* $u_{k_0}, ..., u_k \in \mathbf{R}$ *there exists a Cauchy sequence* $\{r_n\} \subset P$ *in* $W^{k,p}(\Delta, \mu)$ *with* $\lim_{n\to\infty} \|q_n - r_n\|_{W^{k,p}(\Delta,\mu)} = 0$ *and* $r^{(j)}(x_0) = u_j$ *for* $k_0 \leqslant j \leqslant k$. *Consequently* $P^{k,p}(\Delta, \mu + \nu)$ *is isomorphic to* $P^{k,p}(\Delta, \mu) \times \mathbf{R}^N$.

The following result (which will be used in the proof of Theorem 3.3) is an improvement of Theorem 3.1 in [RARP2]. The same arguments used in the proof of Theorem 3.1 in [RARP2] give this result.

THEOREM 3.2. *Let us consider* $1 \leqslant p < \infty$, $\mu = (\mu_0, ..., \mu_k)$ *a vectorial measure and a closed set* $I \subseteq \Delta$ *with* $\mu|_I$ *p-admissible and* $(I, \mu) \in \mathscr{C}_0$. *Assume that* $K \subseteq I$ *is a finite union of compact intervals* $J_1, ..., J_n$ *and that for every* J_m *there is an integer* $0 \leqslant k_m \leqslant k$ *satisfying* $J_m \subseteq \Omega^{(k_m - 1)}$, *if* $k_m > 0$, *and* $\mu_j(J_m) = 0$ *for* $k_m < j \leqslant k$, *if* $k_m < k$. *If* $\mu_j(K) < \infty$ *for* $0 < j \leqslant k$, *then there exists a positive constant* c_0 *such that*

$$c_0 \|fg\|_{W^{k,p}(\Delta,\mu)} \leqslant \|f\|_{W^{k,p}(I,\mu)} \Big(\sup_{x\in I} |g(x)| + \|g\|_{W^{k,p}(I,\mu)}\Big),$$

for every $f, g \in V^{k,p}(I, \mu)$ *and defined on* Δ *with* $\operatorname{supp}(fg) \subseteq I$ *and* $g' = g'' = \cdots = g^{(k)} = 0$ *in* $I \setminus K$.

Remark. The sets $\Omega^{(j)}$ are constructed with respect to (I, μ).

Theorem 3.2 gives the following result corresponding to the case $n = 1$ and $k_1 = k$.

COROLLARY 3.4. *Let us consider* $1 \leqslant p < \infty$, $\mu = (\mu_0, ..., \mu_k)$ *a vectorial measure and a closed set* $I \subseteq \Delta$ *with* $\mu|_I$ *p-admissible and* $(I, \mu) \in \mathscr{C}_0$. *Assume that* K *is a compact interval contained in* $I \cap \Omega^{(k-1)}$. *If* $\mu_j(K) < \infty$ *for* $0 < j \leqslant k$, *then there exists a positive constant* c_0 *such that*

$$c_0 \|fg\|_{W^{k,p}(\Delta,\mu)} \leqslant \|f\|_{W^{k,p}(I,\mu)} \Big(\sup_{x\in I} |g(x)| + \|g\|_{W^{k,p}(I,\mu)}\Big),$$

for every $f, g \in V^{k,p}(I, \mu)$ *and defined on* Δ *with* $\operatorname{supp}(fg) \subseteq I$ *and* $g' = g'' = \cdots = g^{(k)} = 0$ *in* $I \setminus K$.

We need some technical result.

LEMMA 3.2. *Let us consider* $1 < p < \infty$, $c_1, c_2 > 0$ *and* w *a (one-dimensional) weight.*

(1) *If* w *satisfies*

$$\|w\|_{L^1([\alpha,\beta])} < c_1 \qquad \textit{and} \qquad c_2 < \|w^{-1}\|_{L^{1/(p-1)}([\alpha,\beta])},$$

then there exists a weight $v \geqslant w$ *such that*

$$\|v\|_{L^1([\alpha,\beta])} < c_1 \qquad \text{and} \qquad c_2 < \|v^{-1}\|_{L^{1/(p-1)}([\alpha,\beta])} < \infty .$$

(2) *If* $w \in L^1([a,b])$ *and satisfies*

$$\|w^{-1}\|_{L^{1/(p-1)}([a,a+\varepsilon])} = \infty, \qquad \text{for every} \quad \varepsilon > 0,$$

then there exists a weight $v \geqslant w$ *such that* $v \in L^1([a,b])$,

$$\|v^{-1}\|_{L^{1/(p-1)}([a+\varepsilon,b])} < \infty , \qquad \text{for every} \quad \varepsilon > 0, \text{ and } \|v^{-1}\|_{L^{1/(p-1)}([a,b])} = \infty.$$

Proof. We first prove (1). For each $t > 0$, let us consider the function $w_t := \max(t, w)$, which obviously satisfies $w_t \geqslant w$. Recall that if μ is a σ-finite measure in X, every measurable function $g \geqslant 0$ satisfies

$$\int_X g \, d\mu = \int_0^\infty \mu(\{x \in X : g(x) \geqslant \lambda\}) \, d\lambda.$$

Therefore we have that

$$a(t) := \|w_t\|_{L^1([\alpha,\beta])} = \int_0^\infty |\{x \in [\alpha,\beta] : \max(t, w(x)) \geqslant \lambda\}| \, d\lambda$$

$$= \int_t^\infty |\{x \in [\alpha,\beta] : w(x) \geqslant \lambda\}| \, d\lambda + (\beta - \alpha)\, t,$$

$$b_p(t)^{1/(p-1)} := \int_\alpha^\beta w_t^{-1/(p-1)}$$

$$= \int_0^\infty |\{x \in [\alpha,\beta] : \min(t^{-1/(p-1)}, w(x)^{-1/(p-1)}) \geqslant \lambda\}| \, d\lambda$$

$$= \int_0^{t^{-1/(p-1)}} |\{x \in [\alpha,\beta] : w(x)^{-1/(p-1)} \geqslant \lambda\}| \, d\lambda$$

$$\leqslant t^{-1/(p-1)} (\beta - \alpha) < \infty.$$

Since $a(t)$ and $b_p(t)$ are continuous functions for $t > 0$ and

$$\lim_{t \to 0^+} a(t) = \|w\|_{L^1([\alpha,\beta])}, \qquad \lim_{t \to 0^+} b_p(t) = \|w^{-1}\|_{L^{1/(p-1)}([\alpha,\beta])},$$

we can take $v := w_t$ for small enough $t > 0$.

In order to prove (2), let us choose $x_0 := b$ and $x_{n+1} \in (a, \min\{a+2^{-n}, x_n\}]$ such that

$$\|w^{-1}\|_{L^{1/(p-1)}([x_{n+1}, x_n])} > 1.$$

By part (1) we can take a weight $v_n \geqslant w$ in $[x_{n+1}, x_n]$ with

$$1 < \|v_n^{-1}\|_{L^{1/(p-1)}([x_{n+1}, x_n])} < \infty,$$

and

$$\|v_n\|_{L^1([x_{n+1}, x_n])} \leqslant \|w\|_{L^1([x_{n+1}, x_n])} + x_n - x_{n+1}.$$

If we define v in $(a, b]$ by $v := v_n$ in $(x_{n+1}, x_n]$, we have $v \geqslant w$,

$$\|v^{-1}\|_{L^{1/(p-1)}([a, b])} = \infty, \qquad \|v^{-1}\|_{L^{1/(p-1)}([x_n, b])} < \infty,$$
$$\|v\|_{L^1([a, b])} \leqslant \|w\|_{L^1([a, b])} + b - a,$$

and this finishes the proof. ∎

Theorem A gives that if μ is p-admissible, $(\Delta, \mu) \in \mathscr{C}_0$ and x_0 is $(k-1)$-regular, then we have

$$c_1 \, |f^{(k-1)}(x_0)| \leqslant \|f\|_{W^{k,p}(\Delta, \mu)},$$

for every $f \in W^{k,p}(\Delta, \mu)$. The following result, which will be used to prove Theorem 4.4, says that this inequality is always false if x_0 is not $(k-1)$-regular.

THEOREM 3.3. *Let us consider $1 < p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ a vectorial measure such that there exist $\eta_0 > 0$, $x_0 \in \mathbf{R}$ and $0 < k_0 \leqslant k$ with $\mu_j([x_0 - \eta_0, x_0 + \eta_0]) < \infty$ for $0 \leqslant j \leqslant k_0$ and $\mu_j([x_0 - \eta_0, x_0 + \eta_0]) = 0$ for $k_0 < j \leqslant k$ (if $k_0 < k$). Let us assume that x_0 is neither right nor left $(k_0 - 1)$-regular and that $\mu_{k_0 - 1}(\{x_0\}) = 0$. Then, for any $0 < \eta \leqslant \eta_0$, there is no positive constant c_1 with*

$$c_1 \, |f^{(k_0-1)}(x_0)| \leqslant \|f\|_{W^{k,p}(\Delta, \mu)},$$

for every $f \in C_c^\infty([x_0 - \eta, x_0 + \eta])$.

If we have also that μ is finite and Δ is a compact set, then there is no positive constant c_2 with

$$c_2 \, |q^{(k_0-1)}(x_0)| \leqslant \|q\|_{W^{k,p}(\Delta, \mu)},$$

for every $q \in P$.

Remark. If E is a closed set, we denote by $C_c^\infty(E)$ the set of functions $f \in C_c^\infty(\mathbf{R})$ with supp $f \subseteq E$.

Proof. In order to prove the first part of the theorem, without loss of generality we can assume that $k_0 = k$, since otherwise we can change Δ to $\Delta \cap [x_0 - \eta_0, x_0 + \eta_0]$. Let us denote by w the absolutely continuous part of μ. Observe that the fact that x_0 is neither right nor left $(k-1)$-regular is equivalent to

$$w_k \notin B_p([x_0, x_0+\eta]) \cup B_p([x_0-\eta, x_0]), \qquad \text{for every} \quad \eta > 0.$$

We can assume that $w_j(x) \geqslant 1$ for $0 \leqslant j < k$ if $x \in [x_0-\eta_0, x_0+\eta_0]$ and $w_k(x) \in B_p([x_0-\eta, x_0+\eta] \backslash \{x_0\})$, since otherwise we can change $w_j(x)$ to $\max(w_j(x), 1)$ and $w_k(x)$ according to Lemma 3.2 in $[x_0-\eta_0, x_0+\eta_0]$. This increases the right hand side of the first inequality and does not change the fact that $w_k \notin B_p([x_0, x_0+\eta]) \cup B_p([x_0-\eta, x_0])$ for every $\eta > 0$.

Observe that it is enough to prove the first part of Theorem 3.3 for almost every $\eta \in (0, \eta_0]$ (with respect to Lebesgue measure). Let us fix $0 < \eta \leqslant \eta_0$ with $\mu_k(\{x_0-\eta\}) = \mu_k(\{x_0+\eta\}) = 0$ (the set of η's in $(0, \eta_0]$ which do not satisfy this is at most denumerable since $\mu_k([x_0-\eta_0, x_0+\eta_0]) < \infty$).

Since $w_k \in B_p((x_0, x_0+\eta]) \backslash B_p([x_0, x_0+\eta])$, the function

$$U(t) := \int_{x_0+t}^{x_0+\eta} w_k^{-1/(p-1)}$$

is positive and continuous on $(0, \eta)$ and $\lim_{t \to 0^+} U(t) = \infty$; since for any sequence $\{y_n\}$ with $y_n \searrow 0$ as $n \to \infty$

$$\lim_{n \to \infty} \int_{x_0-\eta}^{x_0-y_n} w_k^{-1/(p-1)} = \infty,$$

for n large enough there exists a point $x_n \in (0, \eta)$ such that

$$\int_{x_0+x_n}^{x_0+\eta} w_k^{-1/(p-1)} = \int_{x_0-\eta}^{x_0-y_n} w_k^{-1/(p-1)}. \tag{3.6}$$

We have also $x_n \searrow 0$ as $n \to \infty$. Therefore, we can choose decreasing sequences $\{y_n\}$ and $\{x_n\}$ satisfying (3.6) and $\mu_k(\{x_0 - y_n\}) = \mu_k(\{x_0 + x_n\}) = 0$ for every n.

Let us define $S := \operatorname{supp}(\mu_k)_s$ and

$$h_n := w_k^{-1/(p-1)} (\chi_{[x_0-\eta, x_0-y_n] \backslash S} - \chi_{[x_0+x_n, x_0+\eta] \backslash S}).$$

Observe that $h_n \in L^1(\mathbf{R})$, since $w_k \in B_p([x_0-\eta_0, x_0+\eta_0] \backslash \{x_0\})$.

If we define

$$g_n(x) := \int_{x_0+\eta}^{x} h_n(t)\, \frac{(x-t)^{k-1}}{(k-1)!}\, dt,$$

then $g_n^{(k-1)} = \int_{x_0+\eta}^{x} h_n \in AC_{loc}(\mathbf{R})$. We have also

$$g_n(x) = \int_{x_0+\eta}^{x} g_n^{(k-1)}(t)\, \frac{(x-t)^{k-2}}{(k-2)!}\, dt,$$

$$g_n^{(j)}(x) = \int_{x_0+\eta}^{x} g_n^{(k-1)}(t)\, \frac{(x-t)^{k-j-2}}{(k-j-2)!}\, dt,$$

for $0 \leqslant j \leqslant k-2$. Therefore there exists a positive constant c such that

$$\begin{aligned} \| g_n^{(j)} \|_{L^p([x_0-\eta,\, x_0+\eta],\, \mu_j)} &= \left(\int_{x_0-\eta}^{x_0+\eta} \left| \int_{x_0+\eta}^{x} g_n^{(k-1)}(t)\, \frac{(x-t)^{k-j-2}}{(k-j-2)!}\, dt \right|^p d\mu_j(x) \right)^{1/p} \\ &\leqslant c\, \| g_n^{(k-1)} \|_{L^1([x_0-\eta,\, x_0+\eta])}, \end{aligned} \tag{3.7}$$

for $0 \leqslant j \leqslant k-2$, since $\mu_j([x_0-\eta, x_0+\eta]) < \infty$ for $0 \leqslant j \leqslant k$.

Since $\mu_k(\{x_0-\eta\}) = \mu_k(\{x_0+\eta\}) = \mu_k(\{x_0-y_n\}) = \mu_k(\{x_0+x_n\}) = 0$ and $\mu_k([x_0-\eta, x_0+\eta]) < \infty$, given any $\varepsilon > 0$ we can choose a function $I_n \in C_c((x_0-\eta, x_0-y_n) \cup (x_0+x_n, x_0+\eta))$ such that

$$\| I_n - h_n \|_{L^p([x_0-\eta,\, x_0+\eta],\, \mu_k)} \leqslant \varepsilon \qquad \text{and} \qquad \| I_n - h_n \|_{L^1([x_0-\eta,\, x_0+\eta])} \leqslant \varepsilon,$$

by Lemma 3.1 in [R] (recall that $h_n \in L^p([x_0-\eta,\ x_0+\eta], \mu_k) \cap L^1([x_0-\eta, x_0+\eta])$ and $h_n = 0$ on (x_0-y_n, x_0+x_n)). (This Lemma is just a version of the classical approximation result.) Since $\mu_k([x_0-\eta, x_0+\eta]) < \infty$ and $I_n \in C_c((x_0-\eta, x_0-y_n) \cup (x_0+x_n, x_0+\eta))$, by a convolution of I_n with an approximation of identity, we can find a function $H_n \in C_c^\infty((x_0-\eta, x_0-y_n) \cup (x_0+x_n, x_0+\eta))$ such that

$$\| I_n - H_n \|_{L^p([x_0-\eta,\, x_0+\eta],\, \mu_k)} \leqslant \varepsilon \qquad \text{and} \qquad \| I_n - H_n \|_{L^1([x_0-\eta,\, x_0+\eta])} \leqslant \varepsilon.$$

Then we have

$$\| H_n - h_n \|_{L^p([x_0-\eta,\, x_0+\eta],\, \mu_k)} \leqslant 2\varepsilon \qquad \text{and} \qquad \| H_n - h_n \|_{L^1([x_0-\eta,\, x_0+\eta])} \leqslant 2\varepsilon. \tag{3.8}$$

We now define

$$G_n(x) := \int_{x_0+\eta}^{x} H_n(t)\, \frac{(x-t)^{k-1}}{(k-1)!}\, dt.$$

Let us fix a function $\varphi \in C^\infty(\mathbf{R})$ satisfying $0 \leqslant \varphi \leqslant 1$ in $\mathbf{R}$, $\varphi = 1$ in $[x_0 - \eta/2, \infty)$ and $\varphi = 0$ in $(-\infty, x_0 - \eta]$, and define $F_n := G_n \varphi$.

Assume that there is a positive constant c_1 with

$$c_1 \, |f^{(k-1)}(x_0)| \leqslant \| f \|_{W^{k,p}(\Delta, \mu)},$$

for every $f \in C_c^\infty([x_0 - \eta, x_0 + \eta])$. By Remark 1 after Definition 11 we have $([x_0 - \eta, x_0 + \eta], \mu) \in \mathscr{C}_0$, since $w_k \in B_p([x_0 - \eta, x_0 + \eta] \backslash \{x_0\})$ and $w_j(x) \geqslant 1$ for $0 \leqslant j < k$, if $x \in [x_0 - \eta, x_0 + \eta]$, and this implies that $\Omega^{(0)} \backslash (\Omega_1 \cup \cdots \cup \Omega_k)$ has at most three points ($\{x_0 - \eta, x_0, x_0 + \eta\}$) and $\mathscr{K}([x_0 - \eta, x_0 + \eta], \mu) = \{0\}$. By Corollary 3.4, with $K := [x_0 - \eta, x_0 - \eta/2]$, we have

$$\begin{aligned} c_1 \, |G_n^{(k-1)}(x_0)| = c_1 \, |F_n^{(k-1)}(x_0)| &\leqslant \| F_n \|_{W^{k,p}(\Delta, \mu)} \\ &\leqslant c \, \| G_n \|_{W^{k,p}([x_0 - \eta, x_0 + \eta], \mu)}, \end{aligned}$$

and consequently

$$|G_n^{(k-1)}(x_0)| \leqslant c \, \| G_n \|_{W^{k,p}([x_0 - \eta, x_0 + \eta], \mu)}, \tag{3.9}$$

for every n. In order to apply Corollary 3.4 μ must be p-admissible; otherwise, applying Corollary 3.4, we can obtain (3.9) for μ^{ad} instead of μ (see Definition 15 in Section 4), and we have $\mu^{ad} \leqslant \mu$.

By (3.8), we have that there exists a positive constant c, independent of n and ε, such that

$$\begin{aligned} &\| g_n^{(j)} - G_n^{(j)} \|_{L^p([x_0 - \eta, x_0 + \eta], \mu_j)} \\ &\quad \leqslant \left(\int_{x_0 - \eta}^{x_0 + \eta} \left(\int_x^{x_0 + \eta} |h_n(t) - H_n(t)| \, \frac{|x - t|^{k-j-1}}{(k-j-1)!} \, dt \right)^p d\mu_j(x) \right)^{1/p} \\ &\quad \leqslant c \, \| h_n - H_n \|_{L^1([x_0 - \eta, x_0 + \eta])} \leqslant 2c\varepsilon, \end{aligned}$$

for $0 \leqslant j < k$, since $\mu_j([x_0 - \eta, x_0 + \eta]) < \infty$ for $0 \leqslant j \leqslant k$. This inequality and (3.8) show that there exists a positive constant c such that

$$\| g_n - G_n \|_{W^{k,p}([x_0 - \eta, x_0 + \eta], \mu)} \leqslant c\varepsilon, \tag{3.10}$$

if we choose h_n as $g_n^{(k)}$ (observe that if we change $g_n^{(k)}$ in a set B of zero Lebesgue measure, this would change $\| g_n^{(k)} \|_{L^p([x_0 - \eta, x_0 + \eta], \mu_k)}$ if $\mu_k(B) > 0$).

We have also by (3.8)

$$\begin{aligned} |g_n^{(k-1)}(x_0) - G_n^{(k-1)}(x_0)| &\leqslant \int_{x_0}^{x_0 + \eta} |h_n(t) - H_n(t)| \, dt \\ &\leqslant \| h_n - H_n \|_{L^1([x_0 - \eta, x_0 + \eta])} \leqslant 2\varepsilon. \end{aligned}$$

Therefore, by (3.9) and (3.10), we obtain for some positive constant c

$$\begin{aligned}|g_n^{(k-1)}(x_0)| - 2\varepsilon \leqslant |G_n^{(k-1)}(x_0)| \leqslant c \, \|G_n\|_{W^{k,p}([x_0-\eta, x_0+\eta], \mu)} \\ \leqslant c(\|g_n\|_{W^{k,p}([x_0-\eta, x_0+\eta], \mu)} + c\varepsilon),\end{aligned}$$

for every n and $\varepsilon > 0$. Consequently

$$|g_n^{(k-1)}(x_0)| \leqslant c \, \|g_n\|_{W^{k,p}([x_0-\eta, x_0+\eta], \mu)},$$

for every n. Therefore by (3.7) we have that there exists a positive constant c such that

$$\begin{aligned}c \, |g_n^{(k-1)}(x_0)| \leqslant \|g_n^{(k-1)}\|_{L^1([x_0-\eta, x_0+\eta])} + \|g_n^{(k-1)}\|_{L^p([x_0-\eta, x_0+\eta], \mu_{k-1})} \\ + \|g_n^{(k)}\|_{L^p([x_0-\eta, x_0+\eta], \mu_k)},\end{aligned}$$

for every n. Since $w_{k-1} \geqslant 1$ in $[x_0-\eta, x_0+\eta]$, there exists a positive constant c such that

$$\begin{aligned}c \, |g_n^{(k-1)}(x_0)| &\leqslant \|g_n^{(k-1)}\|_{L^p([x_0-\eta, x_0+\eta], \mu_{k-1})} + \|h_n\|_{L^p([x_0-\eta, x_0+\eta], \mu_k)} \\ &= \|g_n^{(k-1)}\|_{L^p([x_0-\eta, x_0+\eta], \mu_{k-1})} + \|h_n\|_{L^p([x_0-\eta, x_0+\eta], w_k)},\end{aligned}$$

for every n, since $g_n^{(k)} = h_n = 0$ in $S = \operatorname{supp}(\mu_k)_s$.

For each $\varepsilon > 0$ there exists $\delta > 0$ with $\mu_{k-1}([x_0-\delta, x_0+\delta]) < \varepsilon$, since $\mu_{k-1}(\{x_0\}) = 0$ and $\mu_{k-1}([x_0-\eta, x_0+\eta])$ is finite. Recall that $g_n^{(k-1)} \in AC([x_0-\eta, x_0+\eta])$. Therefore, we have that

$$\begin{aligned}c \, |g_n^{(k-1)}(x_0)| \leqslant{} & \varepsilon^{1/p} \, \|g_n^{(k-1)}\|_{L^\infty([x_0-\delta, x_0+\delta])} \\ & + \|g_n^{(k-1)}\|_{L^p([x_0-\eta, x_0-\delta] \cup [x_0+\delta, x_0+\eta], \mu_{k-1})} \\ & + \|h_n\|_{L^p([x_0-\eta, x_0+\eta], w_k)} \\ ={} & \varepsilon^{1/p} \, |g_n^{(k-1)}(x_0)| + \|g_n^{(k-1)}\|_{L^p([x_0-\eta, x_0-\delta] \cup [x_0+\delta, x_0+\eta], \mu_{k-1})} \\ & + (2 \, |g_n^{(k-1)}(x_0)|)^{1/p}, \qquad (3.11)\end{aligned}$$

since $g_n^{(k-1)}(x) = \int_{x_0+\eta}^x h_n$, $\|g_n^{(k-1)}\|_{L^\infty([x_0-\delta, x_0+\delta])} = |g_n^{(k-1)}(x_0)|$, and (3.6) shows

$$\begin{aligned}\|h_n\|^p_{L^p([x_0-\eta, x_0+\eta], w_k)} &= \int_{x_0-\eta}^{x_0-y_n} w_k^{-1/(p-1)} + \int_{x_0+x_n}^{x_0+\eta} w_k^{-1/(p-1)} \\ &= 2 \, |g_n^{(k-1)}(x_0)|.\end{aligned}$$

Since $g_n^{(k-1)}(x) = \int_{x_0+\eta}^x h_n$ and $g_n^{(k-1)}(x_0) = \int_{x_0+x_n}^{x_0+\eta} w_k^{-1/(p-1)}$, we have that $\lim_{n\to\infty} |g_n^{(k-1)}(x_0)| = \int_{x_0}^{x_0+\eta} w_k^{-1/(p-1)} = \infty$, since $w_k \notin B_p([x_0, x_0+\eta])$.

Claim. We have that $\|g_n^{(k-1)}\|_{L^p([x_0-\eta,x_0-\delta]\cup[x_0+\delta,x_0+\eta],\mu_{k-1})}$ is bounded.

If we have the claim, then as $n\to\infty$ in (3.11), we obtain $c\leqslant\varepsilon^{1/p}$ (recall that $\lim_{n\to\infty}|g_n^{(k-1)}(x_0)|=\infty$), and since $\varepsilon>0$ is arbitrary we conclude that $c=0$, which is a contradiction. This finishes the proof of the first part of Theorem 3.3, except for the claim.

We now prove the claim. We have for $x\in[x_0+\delta,x_0+\eta]$

$$0\leqslant g_n^{(k-1)}(x)\leqslant\int_{x_0+\delta}^{x_0+\eta} w_k^{-1/(p-1)}.$$

The fact (3.6) gives $g_n^{(k-1)}(x_0-\eta)=0$, and therefore $g_n^{(k-1)}(x)=\int_{x_0-\eta}^{x} h_n$. Then we have for $x\in[x_0-\eta,x_0-\delta]$

$$0\leqslant g_n^{(k-1)}(x)\leqslant\int_{x_0-\eta}^{x_0-\delta} w_k^{-1/(p-1)}.$$

This finishes the proof of the claim.

If we have also that μ is finite and Δ is a compact set, then we obtain the result for polynomials, since we can approximate the kth derivative of each function in $C^k(\mathbf{R})$ uniformly in Δ by polynomials. ∎

Theorems 3.3 and A give the following result.

COROLLARY 3.5. *Let us consider $1<p<\infty$ and $\mu=(\mu_0,...,\mu_k)$ a p-admissible vectorial measure with $(\Delta,\mu)\in\mathscr{C}_0$ and such that there exist $\eta_0>0$, $x_0\in\mathbf{R}$ and $0<k_0\leqslant k$ with $\mu_j([x_0-\eta_0,x_0+\eta_0])<\infty$ for $0\leqslant j\leqslant k_0$ and $\mu_j([x_0-\eta_0,x_0+\eta_0])=0$ for $k_0<j\leqslant k$ (if $k_0<k$). Then, there is a positive constant c_1 with*

$$c_1\,|f^{(k_0-1)}(x_0)|\leqslant\|f\|_{W^{k,p}(\Delta,\mu)},$$

for every $f\in C_c^\infty([x_0-\eta_0,x_0+\eta_0])$ if and only if x_0 is right or left (k_0-1)-regular.

4. PROOF OF THE RESULTS FOR M

First of all, some remarks about the definition of the multiplication operator.

DEFINITION 12. We say that the multiplication operator is well defined in $P^{k,p}(\Delta,\mu)$ if given any sequence $\{s_n\}$ of polynomials converging to 0 in $W^{k,p}(\Delta,\mu)$, then $\{xs_n\}$ also converges to 0 in $W^{k,p}(\Delta,\mu)$. In this case, if

$\{q_n\} \in P^{k,p}(\Delta, \mu)$, we define $M(\{q_n\}) := \{xq_n\}$. If we choose another Cauchy sequence $\{r_n\}$ representing the same element in $P^{k,p}(\Delta, \mu)$ (i.e. $\{q_n - r_n\}$ converges to 0 in $W^{k,p}(\Delta, \mu)$), then $\{xq_n\}$ and $\{xr_n\}$ represent the same element in $P^{k,p}(\Delta, \mu)$ (since $\{x(q_n - r_n)\}$ converges to 0 in $W^{k,p}(\Delta, \mu)$).

This definition is as natural as the following.

DEFINITION 13. *If μ is a p-admissible vectorial measure (and hence $W^{k,p}(\Delta, \mu)$ is a space of classes of functions), we say that the multiplication operator is well defined in $W^{k,p}(\Delta, \mu)$ if given any function $h \in V^{k,p}(\Delta, \mu)$ with $\|h\|_{W^{k,p}(\Delta,\mu)} = 0$, we have $\|xh\|_{W^{k,p}(\Delta,\mu)} = 0$. In this case, if $[f]$ is an equivalence class in $W^{k,p}(\Delta, \mu)$, we define $M([f]) := [xf]$. If we choose another representative g of $[f]$ (i.e., $\|f - g\|_{W^{k,p}(\Delta,\mu)} = 0$) we have $[xf] = [xg]$, since $\|x(f-g)\|_{W^{k,p}(\Delta,\mu)} = 0$.*

The following result characterizes the spaces $W^{k,p}(\Delta, \mu)$ with M well defined in the sense of Definition 13 [RARP2, Theorem 5.2].

THEOREM C. *Let us consider $1 \leqslant p < \infty$ and a p-admissible vectorial measure μ. Assume that $xf \in V^{k,p}(\Delta, \mu)$ for every $f \in V^{k,p}(\Delta, \mu)$. Then M is well defined in $W^{k,p}(\Delta, \mu)$ if and only if $\mathscr{K}(\Delta, \mu) = \{0\}$.*

Although both definitions are natural, it is possible for a p-admissible measure μ with $W^{k,p}(\Delta, \mu) = \bar{P}$ (the closure of P is considered with the norm in $W^{k,p}(\Delta, \mu)$) that M is well defined in $W^{k,p}(\Delta, \mu)$ and not well defined in $P^{k,p}(\Delta, \mu)$ (see Corollary 4.4). The following lemma characterizes the spaces $P^{k,p}(\Delta, \mu)$ with M well defined.

Remark. From now on we use Definition 12 instead of Definition 13.

LEMMA 4.1. *Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ a vectorial measure in $\mathscr{M}$. The following facts are equivalent:*

(1) *The multiplication operator is well defined in $P^{k,p}(\Delta, \mu)$.*

(2) *The multiplication operator is bounded in $P^{k,p}(\Delta, \mu)$.*

(3) *There exists a positive constant c such that*

$$\|xq\|_{W^{k,p}(\Delta,\mu)} \leqslant c\,\|q\|_{W^{k,p}(\Delta,\mu)}, \qquad \textit{for every} \quad q \in P.$$

Remark. When we say that the multiplication operator is bounded in $P^{k,p}(\Delta, \mu)$, we are assuming implicitly that it is well defined in $P^{k,p}(\Delta, \mu)$, since otherwise the boundedness has no sense.

Proof. It is clear that condition (3) implies (1). If we assume (1), we have that the multiplication operator M is continuous in $0 \in (P, \|\cdot\|_{W^{k,p}(\Delta,\mu)})$.

Since M is a linear operator in the normed space $(P, \|\cdot\|_{W^{k,p}(\Delta,\mu)})$, we know that M is bounded in $(P, \|\cdot\|_{W^{k,p}(\Delta,\mu)})$, which gives (3).

We now show the equivalence between (2) and (3). Let us consider an element $\gamma \in P^{k,p}(\Delta,\mu)$. This element γ is an equivalence class of Cauchy sequences of polynomials under the norm in $W^{k,p}(\Delta,\mu)$. Assume that a Cauchy sequence of polynomials $\{q_n\}$ represents γ. The norm of γ is defined as $\|\gamma\|_{P^{k,p}(\Delta,\mu)} = \lim_{n\to\infty} \|q_n\|_{W^{k,p}(\Delta,\mu)}$, which obviously does not depend on the representative chosen. Hence, condition (2) is equivalent to

$$\lim_{n\to\infty} \|xq_n\|_{W^{k,p}(\Delta,\mu)} \leqslant c \lim_{n\to\infty} \|q_n\|_{W^{k,p}(\Delta,\mu)},$$

for every Cauchy sequence of polynomials $\{q_n\}$. Now the equivalence between (2) and (3) is clear. ∎

We now deduce the following particular case.

Corollary 4.1. *Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ a p-admissible vectorial measure in $\mathscr{M}$ with $W^{k,p}(\Delta,\mu) = \overline{P}$. If the multiplication operator is well defined in $P^{k,p}(\Delta,\mu)$, then it is well defined and bounded in $W^{k,p}(\Delta,\mu)$.*

Lemma 4.2. *Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ a finite vectorial measure with Δ a compact set. Then, the multiplication operator is bounded in $P^{k,p}(\Delta,\mu)$ if and only if there exists a positive constant c such that*

$$\|q^{(j-1)}\|_{L^p(\Delta,\mu_j)} \leqslant c\, \|q\|_{W^{k,p}(\Delta,\mu)},$$

for every $1 \leqslant j \leqslant k$ and $q \in P$.

Proof. If M is bounded in $P^{k,p}(\Delta,\mu)$, we have that

$$\|(xq)^{(j)}\|_{L^p(\Delta,\mu_j)} \leqslant \|M\|\, \|q\|_{W^{k,p}(\Delta,\mu)},$$

for every $1 \leqslant j \leqslant k$ and $q \in P$. Since

$$\begin{aligned} \|(xq)^{(j)}\|_{L^p(\Delta,\mu_j)} &= \|xq^{(j)} + jq^{(j-1)}\|_{L^p(\Delta,\mu_j)} \\ &\geqslant \|q^{(j-1)}\|_{L^p(\Delta,\mu_j)} - K\, \|q^{(j)}\|_{L^p(\Delta,\mu_j)}, \end{aligned}$$

with $K := \max\{|x| : x \in \Delta\}$, we have

$$\begin{aligned}\|q^{(j-1)}\|_{L^p(\Delta,\mu_j)} &\leqslant K\,\|q^{(j)}\|_{L^p(\Delta,\mu_j)} + \|M\|\,\|q\|_{W^{k,p}(\Delta,\mu)}\\ &\leqslant (K+\|M\|)\,\|q\|_{W^{k,p}(\Delta,\mu)},\end{aligned}$$

for every $1 \leqslant j \leqslant k$ and $q \in P$.

We now prove the converse implication. We have

$$\begin{aligned}\|(xq)^{(j)}\|_{L^p(\Delta,\mu_j)} &= \|xq^{(j)} + jq^{(j-1)}\|_{L^p(\Delta,\mu_j)}\\ &\leqslant j\,\|q^{(j-1)}\|_{L^p(\Delta,\mu_j)} + K\,\|q^{(j)}\|_{L^p(\Delta,\mu_j)},\end{aligned}$$

with $K := \max\{|x| : x \in \Delta\}$, for every $1 \leqslant j \leqslant k$ and $q \in P$. Then

$$\begin{aligned}\|(xq)^{(j)}\|^p_{L^p(\Delta,\mu_j)} &\leqslant 2^{p-1}(\,j^p\,\|q^{(j-1)}\|^p_{L^p(\Delta,\mu_j)} + K^p\,\|q^{(j)}\|^p_{L^p(\Delta,\mu_j)})\\ &\leqslant 2^{p-1}(k^p c^p\,\|q\|^p_{W^{k,p}(\Delta,\mu)} + K^p\,\|q^{(j)}\|^p_{L^p(\Delta,\mu_j)}),\end{aligned}$$

for every $0 \leqslant j \leqslant k$ and $q \in P$ (if $j=0$ the inequality is trivial). Consequently

$$\|xq\|^p_{W^{k,p}(\Delta,\mu)} \leqslant 2^{p-1}(k^{p+1}c^p\,\|q\|^p_{W^{k,p}(\Delta,\mu)} + K^p\,\|q\|^p_{W^{k,p}(\Delta,\mu)}),$$

and

$$\|xq\|_{W^{k,p}(\Delta,\mu)} \leqslant 2^{(p-1)/p}(k^{p+1}c^p + K^p)^{1/p}\,\|q\|_{W^{k,p}(\Delta,\mu)},$$

for every $q \in P$. Hence, Lemma 4.1 gives that M is bounded in $P^{k,p}(\Delta,\mu)$. ∎

In the following we often use the next result. We omit the proof since it is elementary.

Lemma 4.3. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, ..., \mu_k)$, $\mu' = (\mu'_0, ..., \mu'_k)$ *vectorial measures in* $\mathcal{M}$, *with* $\Delta = \bigcup_{j=0}^k \operatorname{supp} \mu_j = \bigcup_{j=0}^k \operatorname{supp} \mu'_j$. *If the Sobolev norms in* $W^{k,p}(\Delta,\mu)$ *and* $W^{k,p}(\Delta,\mu')$ *are comparable on* P, *then*:

(1) $P^{k,p}(\Delta,\mu) = P^{k,p}(\Delta,\mu')$.

(2) *M is bounded in $P^{k,p}(\Delta,\mu)$ if and only if it is bounded in $P^{k,p}(\Delta,\mu')$.*

Definition 14. *We say that a vectorial measure* $\mu = (\mu_0, ..., \mu_k)$ *belongs to the class ESD if* $d\mu_j = f_j\, d\mu_{j-1}$, *with* f_j *bounded for* $1 \leqslant j \leqslant k$.

Remark. A vectorial measure μ is sequentially dominated if and only if $\mu \in ESD$ and $\#\operatorname{supp}\mu_0 = \infty$. If $\mu \in ESD$, observe that 0 is the unique polynomial q with $\|q\|_{W^{k,p}(\Delta,\mu)} = 0$ if and only if $\#\operatorname{supp}\mu_0 = \infty$.

THEOREM 4.1. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, ..., \mu_k)$ *a finite vectorial measure with* Δ *a compact set. Then, the multiplication operator is bounded in* $P^{k,p}(\Delta, \mu)$ *if and only if there exists a vectorial measure* $\mu' \in ESD$ *such that the Sobolev norms in* $W^{k,p}(\Delta, \mu)$ *and* $W^{k,p}(\Delta, \mu')$ *are comparable on* P. *Furthermore, we can choose* $\mu' = (\mu'_0, ..., \mu'_k)$ *with* $\mu'_j := \mu_j + \mu_{j+1} + \cdots + \mu_k$.

Proof. Assume that there exists a vectorial measure $\mu' \in ESD$ such that the Sobolev norms in $W^{k,p}(\Delta, \mu)$ and $W^{k,p}(\Delta, \mu')$ are comparable on P. By lemmas 4.2 and 4.3 it is enough to show

$$\|q^{(j-1)}\|_{L^p(\Delta, \mu'_j)} \leqslant c \, \|q\|_{W^{k,p}(\Delta, \mu')}, \tag{4.1}$$

for every $1 \leqslant j \leqslant k$ and $q \in P$. The hypothesis $\mu' \in ESD$ gives

$$\int |q^{(j-1)}|^p \, d\mu'_j = \int |q^{(j-1)}|^p \, f_j \, d\mu'_{j-1} \leqslant \|f_j\|_\infty \int |q^{(j-1)}|^p \, d\mu'_{j-1},$$

where $\|f_j\|_\infty = \sup_{x \in \Delta} |f_j(x)|$, and then we have (4.1).

Assume now that M is bounded in $P^{k,p}(\Delta, \mu)$. Let us consider the vectorial measures $\mu^0, \mu^1, ..., \mu^{k-1}, \mu^k$ defined by

$$\mu_i^j := \mu_i, \qquad \text{if} \quad 0 \leqslant i < j,$$

$$\mu_i^j := \sum_{l=i}^{k} \mu_l, \qquad \text{if} \quad j \leqslant i \leqslant k.$$

Observe that $\mu^k = \mu$ and μ^0 is the measure μ' defined at the end of the statement of Theorem 4.1. These vectorial measures satisfy, for $0 \leqslant i \leqslant k$ and $0 < j \leqslant k$,

$$\mu_i^{j-1} := \mu_i^j, \qquad \text{if} \quad i \neq j-1, \tag{4.2}$$

$$\mu_{j-1}^{j-1} := \mu_j^j + \mu_{j-1} = \mu_j^j + \mu_{j-1}^j. \tag{4.3}$$

Therefore we have $\|q\|_{W^{k,p}(\Delta, \mu^j)} \leqslant \|q\|_{W^{k,p}(\Delta, \mu^{j-1})}$, for every $q \in P$ and $1 \leqslant j \leqslant k$.

Since $\mu^0 \in ESD$ it is enough to show that the Sobolev norms in $W^{k,p}(\Delta, \mu^k)$ and $W^{k,p}(\Delta, \mu^0)$ are comparable on P. We prove this by showing for $1 \leqslant j \leqslant k$ that the Sobolev norms in $W^{k,p}(\Delta, \mu^j)$ and $W^{k,p}(\Delta, \mu^{j-1})$ are comparable on P and M is bounded in $P^{k,p}(\Delta, \mu^{j-1})$. We prove this last statement by reverse induction on j.

If $j = k$, we have that M is bounded in $P^{k,p}(\Delta, \mu^k)$, since $\mu^k = \mu$. Lemma 4.2 gives that

$$\|q^{(k-1)}\|_{L^p(\Delta, \mu_k^k)} = \|q^{(k-1)}\|_{L^p(\Delta, \mu_k)} \leqslant c \, \|q\|_{W^{k,p}(\Delta, \mu)} = c \, \|q\|_{W^{k,p}(\Delta, \mu^k)},$$

for every $q \in P$. This inequality and (4.3) give

$$\|q^{(k-1)}\|^p_{L^p(\Delta, \mu^{k-1}_{k-1})} \leqslant c^p \|q\|^p_{W^{k,p}(\Delta, \mu^k)} + \|q^{(k-1)}\|^p_{L^p(\Delta, \mu^k_{k-1})}$$
$$\leqslant (c^p+1) \|q\|^p_{W^{k,p}(\Delta, \mu^k)},$$

for every $q \in P$. This fact and (4.2) show that the Sobolev norms in $W^{k,p}(\Delta, \mu^k)$ and $W^{k,p}(\Delta, \mu^{k-1})$ are comparable on P. Therefore Lemma 4.3 shows that M is bounded in $P^{k,p}(\Delta, \mu^{k-1})$, since it is bounded in $P^{k,p}(\Delta, \mu^k)$.

Assume now that the induction hypothesis holds for $j+1$. Then we have that M is bounded in $P^{k,p}(\Delta, \mu^j)$. Lemma 4.2 shows that

$$\|q^{(j-1)}\|_{L^p(\Delta, \mu^j_j)} \leqslant c \|q\|_{W^{k,p}(\Delta, \mu^j)},$$

for every $q \in P$. This inequality and (4.3) show

$$\|q^{(j-1)}\|^p_{L^p(\Delta, \mu^{j-1}_{j-1})} \leqslant c^p \|q\|^p_{W^{k,p}(\Delta, \mu^j)} + \|q^{(j-1)}\|^p_{L^p(\Delta, \mu^j_{j-1})}$$
$$\leqslant (c^p+1) \|q\|^p_{W^{k,p}(\Delta, \mu^j)},$$

for every $q \in P$. This fact and (4.2) show that the Sobolev norms in $W^{k,p}(\Delta, \mu^j)$ and $W^{k,p}(\Delta, \mu^{j-1})$ are comparable on P. Then Lemma 4.3 shows that M is bounded in $P^{k,p}(\Delta, \mu^{j-1})$, since it is bounded in $P^{k,p}(\Delta, \mu^j)$.

This finishes the induction argument and the proof of Theorem 4.1. ∎

Obviously the best way to deduce that μ and μ' are comparable is to prove that μ' can be obtained by a finite number of completions of μ. In order to check this the following result is useful.

PROPOSITION 4.1. *Let us consider* $1 \leqslant p < \infty$, *a vectorial measure* μ *and a fixed* $0 < j \leqslant k$. *Assume that* $\mu_j((a, b]) < \infty$, $w_j := d\mu_j/dx \in B_p((a, b])$, w_j *is comparable to a monotone function in* $(a, a+\varepsilon]$ *and* $(\mu_j)_s\ ((a, a+\varepsilon]) = 0$ *for some* $\varepsilon > 0$. *Then* $\Lambda_p(\mu_j, \mu_j) < \infty$, *where we are considering the interval* $(a, b]$ *in the definition of* Λ_p.

Remark. The result is not true without the monotonicity hypothesis, even when μ would be absolutely continuous, as is shown in the following example.

EXAMPLE. For each $1 \leqslant p < \infty$ there exists a weight $w \in L^\infty([a, b]) \cap B_p((a, b])$ with $\Lambda_p(w, w) = \infty$:

Without loss of generality we can assume that $[a, b]=[0, 1]$. We give the example for $1<p<\infty$. The case $p=1$ is similar. Choose a sequence of positive numbers $\{h_n\}$ growing to infinity with

$$\lim_{n\to\infty} 2^{-2n}\left(\sum_{j=1}^{n} h_j\, 2^{-2j}\right)^{p-1}=\infty,$$

and define

$$w(x):=\begin{cases} 1, & \text{if} \quad x\in(2^{-2n-1}, 2^{-2n}],\\ h_n^{1-p}, & \text{if} \quad x\in(2^{-2n}, 2^{-2n+1}].\end{cases}$$

It is immediate that

$$\int_0^{2^{-2n}} w\geqslant 2^{-2n-1} \qquad \text{and} \qquad \int_{2^{-2n}}^{1} w^{-1/(p-1)}\geqslant \sum_{j=1}^{n} h_j\, 2^{-2j},$$

and hence we have $\Lambda_p(w, w)=\infty$ and $w\in L^\infty([0, 1])\cap B_p((0, 1])$.

Proof. We prove the case $1<p<\infty$. The proof is similar in the case $p=1$. If $w_j\in B_p([a, b])$ the result is immediate. Assume now that $w_j\notin B_p([a, b])$. Without loss of generality we can assume that w_j is a monotone function in $(a, a+\varepsilon]$. We can assume also that $w_j(a+\varepsilon)<\infty$, since otherwise we can take a smaller ε. Then w_j is a non-decreasing function in $(a, a+\varepsilon]$ and $\lim_{x\to a^+} w_j(x)=0$, since otherwise $w_j\in B_p([a, b])$. For $a<r\leqslant a+\varepsilon$, if $I:=\int_{a+\varepsilon}^{b} w_j^{-1/(p-1)}$, we have that

$$\begin{aligned}\mu_j((a, r])&\left(\int_r^b w_j^{-1/(p-1)}\right)^{p-1}\\ &=\left(\int_a^r w_j\right)\left(\int_r^{a+\varepsilon} w_j^{-1/(p-1)}+I\right)^{p-1}\\ &\leqslant (r-a)\, w_j(r)((a+\varepsilon-r)\, w_j(r)^{-1/(p-1)}+I)^{p-1}\\ &\leqslant \varepsilon\,(\varepsilon+w_j(a+\varepsilon)^{1/(p-1)}\, I)^{p-1}.\end{aligned}$$

For $a+\varepsilon<r<b$,

$$\mu_j((a, r])\left(\int_r^b w_j^{-1/(p-1)}\right)^{p-1}\leqslant \mu_j((a, b])\, I^{p-1}.$$

These two inequalities give $\Lambda_p(\mu_j, \mu_j) < \infty$. ∎

The following result is very useful since in many cases it allows to reduce the study of the boundedness of M in an abstract space $P^{k,p}(\Delta, m)$ to the study of the same property in $P^{k,p}(\Delta, \mu)$ with a p-admissible measure μ. This has two advantages: the new space is a space of functions and there are results of boundedness for M with μ p-admissible (see Theorem 4.3 below and [RARP2, Sect. 5]).

THEOREM 4.2. *For $1 \leqslant p < \infty$, let us consider a finite p-admissible vectorial measure μ and a finite vectorial measure v in the compact set $\Delta := \bigcup_{j=0}^{k} \operatorname{supp} \mu_j$. Assume that $(\Delta, \mu) \in \mathscr{C}_0$ and that $\operatorname{supp} v_j$ is contained in a finite union of compact intervals $K_{j-1} \subseteq \Omega^{(j-1)}$, for each $1 \leqslant j \leqslant k$. (The sets $\Omega^{(j-1)}$ are constructed with respect to μ.) If the multiplication operator is bounded in $P^{k,p}(\Delta, \mu)$, then it is bounded in $P^{k,p}(\Delta, \mu + v)$.*

Proof. We have that there is a positive constant c such that

$$\|xq\|_{W^{k,p}(\Delta,\mu)} \leqslant c\, \|q\|_{W^{k,p}(\Delta,\mu)},$$

for every $q \in P$. Then, it is enough to show that for some positive constant c we have

$$\|xq\|_{W^{k,p}(\Delta,v)} \leqslant c\, \|q\|_{W^{k,p}(\Delta,\mu+v)},$$

for every $q \in P$. Since $(xq)^{(j)} = xq^{(j)} + jq^{(j-1)}$ and

$$\|xq^{(j)}\|_{L^p(\Delta,v_j)} \leqslant K\, \|q^{(j)}\|_{L^p(\Delta,v_j)} \leqslant K\, \|q\|_{W^{k,p}(\Delta,\mu+v)},$$

with $K := \max\{|x| \colon x \in \Delta\}$, it is enough to show that for some positive constant c we have

$$\|q^{(j-1)}\|_{L^p(\Delta,v_j)} \leqslant c\, \|q\|_{W^{k,p}(\Delta,\mu+v)},$$

for every $1 \leqslant j \leqslant k$ and $q \in P$. The hypothesis on $\operatorname{supp} v_j$, the finiteness of v and Theorem A give

$$\begin{aligned}\|q^{(j-1)}\|_{L^p(\Delta,v_j)} &= \|q^{(j-1)}\|_{L^p(K_{j-1},v_j)} \leqslant c\, \|q^{(j-1)}\|_{L^\infty(K_{j-1})} \\ &\leqslant c\, \|q\|_{W^{k,p}(\Delta,\mu)} \leqslant c\, \|q\|_{W^{k,p}(\Delta,\mu+v)},\end{aligned}$$

for every $1 \leqslant j \leqslant k$ and $q \in P$. This finishes the proof of Theorem 4.2. ∎

DEFINITION 15. *Let us consider a vectorial measure μ with absolutely continuous part w. We define the p-admissible part μ^{ad} of μ by $d\mu_j^{ad} := d(\mu_j)_s|_{\Omega^{(j)}} + w_j\, dx$ for $0 \leqslant j \leqslant k$ and $\Delta^{ad} := \bigcup_{j=0}^{k} \operatorname{supp} \mu_j^{ad}$.*

We have the following consequence of Theorem 4.2.

COROLLARY 4.2. *For $1 \leqslant p < \infty$, let us consider a finite vectorial measure μ with $\Delta = \Delta^{ad}$ a compact set. Assume that $(\Delta, \mu^{ad}) \in \mathscr{C}_0$ and that $\operatorname{supp}(\mu_j - \mu_j^{ad})$ is contained in a finite union of compact intervals $K_{j-1} \subseteq \Omega^{(j-1)}$, for each $1 \leqslant j \leqslant k$. If the multiplication operator is bounded in $P^{k,p}(\Delta, \mu^{ad})$, then it is bounded in $P^{k,p}(\Delta, \mu)$.*

Remark. If $\Delta \neq \Delta^{ad}$, we can use localization results, such as Theorems 4.9 and 4.10.

THEOREM 4.3. *Let us consider $1 \leqslant p < \infty$ and a finite vectorial measure μ with Δ a compact set. Assume that $(\Delta^{ad}, \mu^{ad}) \in \mathscr{C}_0$ and that for each $1 \leqslant j \leqslant k$ we have $\mu_j(\Delta \backslash (J_{j-1} \cup K_{j-1})) = 0$, where K_{j-1} is a finite union of compact intervals contained in $\Omega^{(j-1)}$, and J_{j-1} is a measurable set with $d\mu_j = f_j \, d\mu_{j-1}$ in J_{j-1} and f_j bounded. Then the multiplication operator is bounded in $P^{k,p}(\Delta, \mu)$.*

Proof. By Lemma 4.2 it is enough to show that there exists a positive constant c such that

$$\|q^{(j-1)}\|_{L^p(\Delta, \mu_j)} \leqslant c \, \|q\|_{W^{k,p}(\Delta, \mu)},$$

for every $1 \leqslant j \leqslant k$ and $q \in P$. We have by $\mu_j(\Delta) < \infty$ and Theorem A

$$\int_{K_{j-1}} |q^{(j-1)}|^p \, d\mu_j \leqslant c \, \|q^{(j-1)}\|^p_{L^\infty(K_{j-1})} \leqslant c \, \|q\|^p_{W^{k,p}(\Delta^{ad}, \mu^{ad})} \leqslant c \, \|q\|^p_{W^{k,p}(\Delta, \mu)}.$$

The hypothesis on J_{j-1} gives

$$\int_{J_{j-1}} |q^{(j-1)}|^p \, d\mu_j = \int_{J_{j-1}} |q^{(j-1)}|^p \, f_j \, d\mu_{j-1}$$

$$\leqslant c \int_{J_{j-1}} |q^{(j-1)}|^p \, d\mu_{j-1} \leqslant c \, \|q\|^p_{W^{k,p}(\Delta, \mu)}.$$

These two inequalities and

$$\int |q^{(j-1)}|^p \, d\mu_j \leqslant \int_{K_{j-1}} |q^{(j-1)}|^p \, d\mu_j + \int_{J_{j-1}} |q^{(j-1)}|^p \, d\mu_j,$$

give the desired result for every $1 \leqslant j \leqslant k$ and $q \in P$. This finishes the proof of Theorem 4.3. ∎

THEOREM 4.4. *Let us consider $1 < p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ a finite vectorial measure such that Δ is a compact set and there exist $\eta_0 > 0$, $x_0 \in \mathbf{R}$*

and $0<k_0\leqslant k$ *with* $\mu_j([x_0-\eta_0, x_0+\eta_0])=0$ *for* $k_0<j\leqslant k$. *Let us assume that* x_0 *is neither right nor left* (k_0-1)*-regular. If* $\mu_{k_0}(\{x_0\})>0$ *and* $\mu_{k_0-1}(\{x_0\})=0$, *then the multiplication operator is not bounded in* $P^{k,p}(\Delta, \mu)$.

Proof. Assume that the multiplication operator is bounded in $P^{k,p}(\Delta, \mu)$. Lemma 4.2 gives that there exists a positive constant c such that

$$\|q^{(k_0-1)}\|_{L^p(\Delta, \mu_{k0})}\leqslant c\,\|q\|_{W^{k,p}(\Delta, \mu)},$$

for every $q\in P$. Consequently, since $\mu_{k_0}(\{x_0\})>0$, we have that

$$|q^{(k_0-1)}(x_0)|\leqslant c\,\|q\|_{W^{k,p}(\Delta, \mu)},$$

for every $q\in P$, but this is a contradiction with Theorem 3.3. ∎

THEOREM 4.5. *Let us consider* $1\leqslant p<\infty$ *and* $\mu=(\mu_0, ..., \mu_k)$ *a finite vectorial measure with* Δ *a compact set. Assume that there exist* $x_0\in \mathbf{R}$, $c>0$, $0\leqslant k_0<k$ *and an open neighbourhood* U *of* x_0 *such that*

$$d\mu_{j+1}(x)\leqslant c\,|x-x_0|^p\,d\mu_j(x),$$

for $x\in U\backslash\{x_0\}$ *and* $k_0\leqslant j<k$. *If there exists* $i>k_0$ *with* $\mu_i(\{x_0\})>0$ *and* $\mu_{i-1}(\{x_0\})=0$, *then the multiplication operator is not bounded in* $P^{k,p}(\Delta, \mu)$.

Proof. Let us consider the vectorial finite measure $\nu=(\nu_0, ..., \nu_k)$ defined as follows: $\nu_j:=0$ for $0\leqslant j<k_0$ and $\nu_j:=\mu_j|_{\{x_0\}}$ for $k_0\leqslant j\leqslant k$. The measure $\mu':=\mu-\nu$ satisfies $\mu'_{k_0}(\{x_0\})=0$ and

$$d\mu'_{j+1}(x)\leqslant c\,|x-x_0|^p\,d\mu'_j(x),$$

for $x\in U$ and $k_0\leqslant j<k$. Then Theorem 3.1 shows that there exists a sequence of polynomials $\{r_n\}$ such that

(1) $\lim_{n\to\infty} \|r_n\|_{W^{k,p}(\Delta, \mu')}=0$,

(2) $r_n^{(i-1)}(x_0)=1$,

(3) $r_n^{(m)}(x_0)=0$, if $m\neq i-1$, $k_0\leqslant m\leqslant k$.

We have $\lim_{n\to\infty} \|r_n\|_{W^{k,p}(\Delta,\mu)} = 0$, by $\mu_{i-1}(\{x_0\}) = 0$ and conditions (1) and (3); conditions (2) and (3) give that

$$\liminf_{n\to\infty} \|xr_n\|_{W^{k,p}(\Delta,\mu)} \geqslant \mu_i(\{x_0\})^{1/p} \lim_{n\to\infty} |x_0\, r_n^{(i)}(x_0) + i r_n^{(i-1)}(x_0)| = i\mu_i(\{x_0\})^{1/p} > 0.$$

These two facts show that the multiplication operator is not bounded in $P^{k,p}(\Delta,\mu)$. ∎

As a particular case we obtain

COROLLARY 4.3. *Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ a finite vectorial measure with Δ a compact set. Assume that there exist $x_0 \in \mathbf{R}$, an open neighbourhood U of x_0 and $0 \leqslant k_0 < k$ with $\mu_j(U \setminus \{x_0\}) = 0$ for $k_0 < j \leqslant k$. If there exists $i > k_0$ with $\mu_i(\{x_0\}) > 0$ and $\mu_{i-1}(\{x_0\}) = 0$, then the multiplication operator is not bounded in $P^{k,p}(\Delta,\mu)$.*

The following is a modification of the Muckenhoupt inequality, which can be proved by similar arguments.

LEMMA 4.4. *Let us consider $1 \leqslant p < \infty$ and μ_0, μ_1 finite measures in $[a,b]$ with $w_1 := d\mu_1/dx$. Assume that there exists a positive constant c_0 with*

$$\left\| \int_x^b g(t)\, dt \right\|_{L^p([a,b],\mu_0)} \leqslant c_0 \|g\|_{L^p([a,b],\mu_1)},$$

for any function g in $C_c^\infty((a,b))$. Then we have

$$\sup_{a<r<b} \mu_0([a,r])\, \|w_1^{-1}\|_{L^{1/(p-1)}([r,b])} \leqslant c_0^p.$$

THEOREM 4.6. *Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ a finite vectorial measure with Δ a compact set. Assume that there exist $0 < k_0 \leqslant k$, $\alpha \in \mathbf{R}$ and $\varepsilon, c > 0$ with*

$$\text{(1)} \qquad \sup_{\alpha<r<\alpha+\varepsilon} \mu_{k_0}([\alpha,r]) \left\| \left(\frac{d\mu_{k_0}}{dx}\right)^{-1} \right\|_{L^{1/(p-1)}([r,\alpha+\varepsilon])} = \infty,$$

$$\text{(2)} \qquad \left\| \int_x^{\alpha+\varepsilon} g(t)\,(x-t)^{k_0-j-1}\, dt \right\|_{L^p(\Delta,\mu_j)} \leqslant c\, \|g\|_{L^p([\alpha,\alpha+\varepsilon],\mu_{k_0})}, \qquad \forall g \in C_c^\infty((\alpha,\alpha+\varepsilon)) \quad \textit{and} \quad 0 \leqslant j < k_0,$$

$$\text{(3)} \qquad \#\operatorname{supp}(\mu_j|_{(\alpha,\alpha+\varepsilon)}) < \infty, \qquad \textit{for} \quad k_0 < j \leqslant k \quad (\textit{if } k_0 < k).$$

Then the multiplication operator is not bounded in $P^{k,p}(\Delta,\mu)$.

Remark. A similar result holds for $[\alpha-\varepsilon, \alpha]$ instead of $[\alpha, \alpha+\varepsilon]$.

Proof. By Lemma 4.4 and hypothesis (1) we know that there exist functions $g_n \in C_c^\infty((\alpha, \alpha+\varepsilon))$ such that

$$\lim_{n\to\infty} \frac{\|\int_x^{\alpha+\varepsilon} g_n(t)\,dt\|_{L^p([\alpha,\alpha+\varepsilon],\mu_{k_0})}}{\|g_n\|_{L^p([\alpha,\alpha+\varepsilon],\mu_{k_0})}} = \infty.$$

If we define $v=(v_0, ..., v_k)$ by $v_j := \mu_j$ for $0 \leqslant j \leqslant k_0$ and $v_j := \mu_j|_{\Delta\setminus(\alpha,\alpha+\varepsilon)}$ for $k_0 < j \leqslant k$ (if $k_0 < k$), and

$$G_n(x) := \int_{\alpha+\varepsilon}^x g_n(t)\,\frac{(x-t)^{k_0-1}}{(k_0-1)!}\,dt,$$

hypothesis (2) gives

$$\|G_n\|_{W^{k,p}(\Delta,v)} \leqslant \sum_{j=0}^{k_0} \|G_n^{(j)}\|_{L^p(\Delta,\mu_j)} \leqslant c_0\,\|g_n\|_{L^p([\alpha,\alpha+\varepsilon],\mu_{k_0})}.$$

Then we have

$$\liminf_{n\to\infty} \frac{\|G_n^{(k_0-1)}\|_{L^p(\Delta,\mu_{k_0})}}{\|G_n\|_{W^{k,p}(\Delta,v)}} \geqslant \lim_{n\to\infty} \frac{\|\int_x^{\alpha+\varepsilon} g_n(t)\,dt\|_{L^p([\alpha,\alpha+\varepsilon],\mu_{k_0})}}{c_0\,\|g_n\|_{L^p([\alpha,\alpha+\varepsilon],\mu_{k_0})}} = \infty\,.$$

Since Δ is compact, Bernstein's proof of the Weierstrass Theorem shows that we can approximate G_n by polynomials with the norm

$$\|f\|_A := \sum_{j=0}^k \|f^{(j)}\|_{L^\infty(\Delta)}\,,$$

and hence with the norm

$$\|f\|_B := \|f\|_{W^{k,p}(\Delta,v)} + \|f^{(k_0-1)}\|_{L^p(\Delta,\mu_{k_0})}\,,$$

since μ and v are finite. Consequently there exists a sequence of polynomials $\{q_n\}$ with

$$\lim_{n\to\infty} \frac{\|q_n^{(k_0-1)}\|_{L^p(\Delta,\mu_{k_0})}}{\|q_n\|_{W^{k,p}(\Delta,v)}} = \infty\,.$$

If $k_0 < k$, by hypothesis (3) we can consider

$$\{x_1, ..., x_m\} = \bigcup_{j=k_0+1}^k \operatorname{supp}(\mu_j|_{(\alpha,\alpha+\varepsilon)})\,.$$

If we apply Corollary 3.3 m times we obtain that there exists a sequence of polynomials $\{r_n\}$ with

$$\lim_{n \to \infty} \frac{\|r_n^{(k_0-1)}\|_{L^p(\Delta, \mu_{k_0})}}{\|r_n\|_{W^{k,p}(\Delta, \nu)}} = \infty,$$

and $r_n^{(j)}(x_i) = 0$ for $k_0 < j \leqslant k$ and $1 \leqslant i \leqslant m$. Therefore

$$\lim_{n \to \infty} \frac{\|r_n^{(k_0-1)}\|_{L^p(\Delta, \mu_{k_0})}}{\|r_n\|_{W^{k,p}(\Delta, \mu)}} = \infty.$$

Now Lemma 4.2 finishes the proof. ∎

COROLLARY 4.4. *Given $a, b \in \mathbf{R}$, $a < b$ and $1 \leqslant p < \infty$, there exists a p-admissible vectorial measure μ such that P is dense in $W^{k,p}([a,b], \mu)$, M is well defined in $W^{k,p}([a,b], \mu)$ and it is not well defined in $P^{k,p}([a,b], \mu)$.*

Proof. Fix $c \in (a, b)$ and define the absolutely continuous vectorial measure μ as follows: $w_j := 1$ in $(c, b]$ for $0 \leqslant j \leqslant k$, $w_j := 0$ in $[a, c]$ for $0 \leqslant j < k$, and w_k is a weight in $[a, c]$ as in the example after Proposition 4.1. μ is a p-admissible measure since it is absolutely continuous. It is easy to check that $\mathscr{K}([a,b], \mu) = \{0\}$ and then Theorem C shows that M is well defined in $W^{k,p}([a,b], \mu)$. Theorem 4.6 with $k_0 = k$ and $[\alpha, \alpha + \varepsilon] = [a, c]$ shows that M is not bounded in $P^{k,p}([a,b], \mu)$; hence it is not well defined in $P^{k,p}([a,b], \mu)$ by Lemma 4.1. Finally, Theorem 3.1 in [R] shows that P is dense in $W^{k,p}([a,b], \mu)$, since $w_k \in B_p((a,b))$. ∎

We present here a case in which the condition $\mu \in ESD$ is equivalent to M bounded.

THEOREM 4.7. *Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ a finite vectorial measure with Δ a compact set. Assume that $\#\operatorname{supp} \mu_j < \infty$ for $1 \leqslant j \leqslant k$. Then, the multiplication operator is bounded in $P^{k,p}(\Delta, \mu)$ if and only if $\mu \in ESD$.*

Proof. Let us assume that $\mu \notin ESD$. Then there exist $x_0 \in \Delta$ and $0 < i \leqslant k$ such that $\mu_i(\{x_0\}) > 0$ and $\mu_{i-1}(\{x_0\}) = 0$, since $\#\operatorname{supp} \mu_j < \infty$ for $1 \leqslant j \leqslant k$. This hypothesis also shows that there exists a neighbourhood U of x_0 with $\mu_j(U \setminus \{x_0\}) = 0$ for $1 \leqslant j \leqslant k$. Then Corollary 4.3 with $k_0 = 0$ gives that the multiplication operator is not bounded in $P^{k,p}(\Delta, \mu)$.

If $\mu \in ESD$, then the proof of Theorem 4.1 shows that M is bounded in $P^{k,p}(\Delta, \mu)$. ∎

Theorem C shows that the condition $\mathscr{K}(\Delta, \mu) \neq \{0\}$ implies that M is not bounded in $P^{k,p}(\Delta, \mu)$ for p-admissible measures μ with $W^{k,p}(\Delta, \mu) = \bar{P}$. If μ is not p-admissible, we can apply the following result.

THEOREM 4.8. *Let us consider $1 \leqslant p < \infty$, $\mu = (\mu_0, ..., \mu_k)$ a finite vectorial measure with Δ a compact set and a connected component (α, β) of $\Omega_1 \cup \cdots \cup \Omega_k$. Assume that $\mu_0(\{\alpha\}) = \mu_0(\{\beta\}) = 0$, $\mu = \mu^{ad}$ in $[\alpha, \beta]$, $\mathscr{K}([\alpha, \beta], \mu) \neq \{0\}$, and that there exist $c, \varepsilon > 0$ with*

$$d\mu_{j+1}(x) \leqslant c\,|x - \alpha|^p\, d\mu_j(x)\,, \qquad x \in [\alpha - \varepsilon, \alpha],$$

$$d\mu_{j+1}(x) \leqslant c\,|x - \beta|^p\, d\mu_j(x)\,, \qquad x \in [\beta, \beta + \varepsilon],$$

for $0 \leqslant j < k$. Then M is not bounded in $P^{k,p}(\Delta, \mu)$.

Proof. Let us consider a function $q \in \mathscr{K}([\alpha, \beta], \mu)$ which is not identically zero. It is easy to see that $q \in P_{k-1}$ (see the arguments in the proof of Proposition 3.1). Without loss of generality we can assume that $\deg q = \max\{\deg r \colon r \in \mathscr{K}([\alpha, \beta], \mu)\}$. Then we have $\|q\|_{W^{k,p}([\alpha, \beta], \mu)} = 0$ and $\|xq\|_{W^{k,p}([\alpha, \beta], \mu)} > 0$, since $xq \notin \mathscr{K}([\alpha, \beta], \mu)$. The hypotheses and Lemma 3.1 show that there exist $g_n \in C_c^\infty(\mathbf{R})$ with

$$\lim_{n \to \infty} \|g_n\|_{W^{k,p}(\Delta, \mu)} = \|q\|_{W^{k,p}([\alpha, \beta], \mu)} = 0$$

and

$$\|xg_n\|_{W^{k,p}(\Delta, \mu)} \geqslant \|xq\|_{W^{k,p}([\alpha, \beta], \mu)} > 0$$

(it is enough to multiply q by functions in $C_c^\infty(\mathbf{R})$ with value 1 in $[\alpha, \beta]$, as in the proof of Theorem 3.1). The proof is finished since g_n and its derivatives can be approximated uniformly by polynomials. ∎

We also have localization results for the multiplication operator.

THEOREM 4.9. *Let us consider $1 \leqslant p < \infty$ and $\mu = (\mu_0, ..., \mu_k)$ a finite vectorial measure with Δ a compact set. Assume that for every $x_0 \in \Delta$ there exist $\varepsilon, c > 0$ (which can depend on x_0) such that*

$$\|xq\|_{W^{k,p}([x_0 - \varepsilon, x_0 + \varepsilon], \mu)} \leqslant c\,\|q\|_{W^{k,p}([x_0 - \varepsilon, x_0 + \varepsilon], \mu)}\,,$$

for every $q \in P$. Then the multiplication operator is bounded in $P^{k,p}(\Delta, \mu)$.

The proof of this result is immediate. We have a partial converse of Theorem 4.9.

THEOREM 4.10. *Let us consider* $1 \leqslant p < \infty$ *and* $\mu = (\mu_0, ..., \mu_k)$ *a finite vectorial measure with* Δ *a compact set. Assume that there exist* $x_0 \in \Delta$ *and* $\varepsilon_0 > 0$ *with* $\mu|_{[x_0 - \varepsilon_0, x_0 + \varepsilon_0]}$ *p-admissible and* $([x_0 - \varepsilon_0, x_0 + \varepsilon_0], \mu) \in \mathscr{C}_0$. *If the multiplication operator is bounded in* $P^{k,p}(\Delta, \mu)$, *then for each* $0 < \varepsilon < \varepsilon_0$ *there exists* $c > 0$ *such that*

$$\|xq\|_{W^{k,p}([x_0 - \varepsilon, x_0 + \varepsilon], \mu)} \leqslant c \, \|q\|_{W^{k,p}([x_0 - \varepsilon_0, x_0 + \varepsilon_0], \mu)},$$

for every $q \in P$.

Proof. Let us fix $0 < \varepsilon < \varepsilon_0$. We can choose compact intervals $J_1 \subseteq [x_0 - \varepsilon_0, x_0 - \varepsilon]$, $J_2 \subseteq [x_0 + \varepsilon, x_0 + \varepsilon_0]$ and integers $0 \leqslant k_1, k_2 \leqslant k$ satisfying for $m = 1, 2$, $J_m \subseteq \Omega^{(k_m - 1)}$, if $k_m > 0$, and $\mu_j(J_m) = 0$ for $k_m < j \leqslant k$, if $k_m < k$. If $K := J_1 \cup J_2$, we have $\mu_j(K) < \infty$ for $0 \leqslant j \leqslant k$, since μ is finite.

Take a fixed function $\varphi \in C_c^\infty((x_0 - \varepsilon_0, x_0 + \varepsilon_0))$ with $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ in $[x_0 - \varepsilon, x_0 + \varepsilon]$ and $\operatorname{supp} \varphi' \subseteq K$. Theorem 3.2 shows that there exists a positive constant c such that

$$\|\varphi g\|_{W^{k,p}(\Delta, \mu)} \leqslant c \, \|g\|_{W^{k,p}([x_0 - \varepsilon_0, x_0 + \varepsilon_0], \mu)}, \tag{4.4}$$

for every $g \in C^k(\mathbf{R})$. Since Δ is a compact set, Bernstein's proof of the Weierstrass Theorem gives that $\|xg\|_{W^{k,p}(\Delta, \mu)} \leqslant \|M\| \, \|g\|_{W^{k,p}(\Delta, \mu)}$, for every $g \in C^k(\mathbf{R})$. Consequently, $\varphi = 1$ in $[x_0 - \varepsilon, x_0 + \varepsilon]$ and (4.4) give

$$\begin{aligned} \|xq\|_{W^{k,p}([x_0 - \varepsilon, x_0 + \varepsilon], \mu)} &\leqslant \|x\varphi q\|_{W^{k,p}(\Delta, \mu)} \\ &\leqslant \|M\| \, \|\varphi q\|_{W^{k,p}(\Delta, \mu)} \leqslant c \, \|q\|_{W^{k,p}([x_0 - \varepsilon_0, x_0 + \varepsilon_0], \mu)}, \end{aligned}$$

for every $q \in P$. This finishes the proof of Theorem 4.10. ∎

Theorem 4.10 is not true without the hypothesis $([x_0 - \varepsilon_0, x_0 + \varepsilon_0], \mu) \in \mathscr{C}_0$, as is shown by the following example.

EXAMPLE. Let us consider the vectorial measure $\mu = (\mu_0, ..., \mu_k)$ with $d\mu_0 := \chi_{[0,1] \cup [2,3]} dx$, $d\mu_k := \chi_{[0,3]} dx$, and $\mu_j := 0$ for $0 < j < k$ if $k > 1$. Theorems 5.1 and 5.2 in [RARP2] show that M is bounded in $W^{k,p}([0,3], \mu)$, since μ is a measure of type 1 in $[0,3]$ (see Definition 10 in [RARP2]) and $\mathscr{K}([0,3], \mu) = \{0\}$. However, for $q(x) := x^{k-1}$ we have for any $0 < \eta < 1/2$, $\|xq\|_{W^{k,p}([1+\eta, 2-\eta], \mu)} = \|k!\|_{L^p([1+\eta, 2-\eta])} > 0$ and $\|q\|_{W^{k,p}([1,2], \mu)} = 0$.

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